Elliptic Function Based Algorithms to Prove Jacobi Theta Function Relations

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Abstract

In this paper we prove identities involving the classical Jacobi theta functions of the form $\sum c(i_1,i_2,i_3,i_4)\theta_1(z|\tau)^{i_1}\theta_2(z|\tau)^{i_2}\theta_3(z|\tau)^{i_3}\theta_4(z|\tau)^{i_4} = 0$ with $c(i_1,i_2,i_3,i_4) \in \mathbb{K}[\Theta]$, where $\mathbb{K}$ is a computable field and $\Theta := \left\{\theta_1^{(2k+1)}(0|\tau) : k \in \mathbb{N}\right\} \cup \left\{\theta_j^{(2k)}(0|\tau) : k \in \mathbb{N} \text{ and } j = 2,3,4\right\}$. We give two algorithms that solve this problem. The second algorithm is simpler and works in a restricted input class.

Key words: Jacobi theta functions, modular forms, algorithmic zero-recognition, computer algebra, automatic proving of special function identities

1. Introduction

Our ultimate goal is to develop computer-assisted treatment for identities among Jacobi theta functions, namely, to automatize the proving procedures of relations and the discovery of relations.

Let us recall the definition of Jacobi theta functions $\theta_j(z|\tau)$ ($j = 1,\ldots,4$):

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Definition 1.1. (DLMF, 2015, 20.2(i)) Let $\tau \in \mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and $q := e^{\pi i \tau}$, then

$$\theta_1(z,q) := \theta_1(z|\tau) := 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin((2n+1)z),$$
$$\theta_2(z,q) := \theta_2(z|\tau) := 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos((2n+1)z),$$
$$\theta_3(z,q) := \theta_3(z|\tau) := 1+2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz),$$
$$\theta_4(z,q) := \theta_4(z|\tau) := 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2nz).$$

As a first step towards the goal we mentioned in the beginning, in Ye (2017) we provided an algorithm to prove identities involving the derivatives of $\theta_j(z|\tau)$ $(j = 1,2,3,4)$, in particular, involving

$$\theta_j^{(k)} := \theta_j^{(k)}(0|\tau) := \left.\frac{\partial^k \theta_j}{\partial z^k}(z|\tau)\right|_{z=0}, \quad k \in \mathbb{N} := \{0,1,2,\dots\}.$$

For example, Algorithm 5.11 of Ye (2017) can assist us to prove identities like

$$\theta_3^{(4)}\theta_3 - 3(\theta_3'')^2 - 2\theta_3^2\theta_2^4\theta_4^4 = 0$$

from (Rademacher, 1973, (93.22)),

$$\frac{\theta_\alpha^{(5)}}{\theta_1'} - 3\left(\frac{\theta_\alpha''}{\theta_\alpha}\right)^2 + 2\left(\frac{\theta_\alpha''}{\theta_\alpha} - \frac{\theta_\beta''}{\theta_\beta}\right)\left(\frac{\theta_\alpha''}{\theta_\alpha} - \frac{\theta_\gamma''}{\theta_\gamma}\right) = 0$$

from (Rademacher, 1973, (93.7)), where $\alpha, \beta, \gamma = 2,3,4$, and

$$\frac{\theta_1^{(3)}}{\theta_1'} - \frac{\theta_2''}{\theta_2} - \frac{\theta_3''}{\theta_3} - \frac{\theta_4''}{\theta_4} = 0$$

from (Lawden, 1989, p. 22).

More generally, in Ye (2017) we showed that this algorithm can do zero-recognition on any function in $\mathbb{K}[\Theta]$, which is the $\mathbb{K}$-algebra generated by

$$\Theta := \left\{\theta_1^{(2k+1)}(0|\tau) : k \in \mathbb{N}\right\} \cup \left\{\theta_j^{(2k)}(0|\tau) : k \in \mathbb{N} \text{ and } j = 2,3,4\right\},$$

where $\mathbb{K} \subseteq \mathbb{C}$ is an effectively computable field which contains all the complex constants we need (i.e., i, $e^{\pi i/4}$, etc.). The reason why we omit $\theta_1^{(k_1)}(0|\tau)$ when $k_1 \in 2\mathbb{N}$, and omit $\theta_m^{(k_2)}(0|\tau)$ $(m = 2,3,4)$ when $k_2 \in 2\mathbb{N}+1$ is that by Definition 1.1 these are equal to zero.

In this article we extend the function space $\mathbb{K}[\Theta]$ to

$$R_1 := \mathbb{K}[\Theta][\theta_1(z|\tau), \theta_2(z|\tau), \theta_3(z|\tau), \theta_4(z|\tau)],$$

which is the $\mathbb{K}[\Theta]$-algebra generated by $\theta_1(z|\tau), \theta_2(z|\tau), \theta_3(z|\tau)$ and $\theta_4(z|\tau)$. In particular, we solve the following problem algorithmically:

Problem 1.1. Given $f \in R_1$, decide whether $f = 0$.

In order to make the presentation simpler, we use the

Notation. For $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Z}^4$:

$$\theta^\alpha(z) := \theta^\alpha(z|\tau) := \theta_1(z|\tau)^{\alpha_1}\theta_2(z|\tau)^{\alpha_2}\theta_3(z|\tau)^{\alpha_3}\theta_4(z|\tau)^{\alpha_4} \equiv \theta_1(z)^{\alpha_1}\theta_2(z)^{\alpha_2}\theta_3(z)^{\alpha_3}\theta_4(z)^{\alpha_4}.$$

For example, our algorithm can prove identities like

$$\theta_2(0)^2\theta_2(z)^2 - \theta_3(0)^2\theta_3(z)^2 + \theta_4(0)^2\theta_4(z)^2 \equiv 0, \text{ [1]} \tag{1.1}$$

$$\theta_2(0)^2\theta_1(z)^2 + \theta_4(0)^2\theta_3(z)^2 - \theta_3(0)^2\theta_4(z)^2 \equiv 0, \tag{1.2}$$

$$\theta_1(z)^4 + \theta_3(z)^4 - \theta_2(z)^4 - \theta_4(z)^4 \equiv 0, \tag{1.3}$$

from (Whittaker and Watson, 1927, p. 466 and p. 469) and (DLMF, 2015, 20.7).

The algorithm can also be used to prove more complicated identities like the following one which is produced by our method in Section 6.1.2 of Ye (2016).

$$c_1\theta_3(z)^2\theta_4(z)^2 + c_2\theta_4(z)^4 + c_3\theta_3(z)^4 + c_4\theta_1(z)^2\theta_2(z)^2 \equiv 0, \tag{1.4}$$

where

$$\begin{gathered} c_1 := -16\theta_2^5\theta_3^2\theta_4^3 - 4\theta_2\theta_3^6\theta_4^3 - 4\theta_2\theta_3^2\theta_4^7 - 32\theta_3^2\theta_4^3\theta_2'' + 32\theta_2\theta_3^2\theta_4^2\theta_4'', \\ c_2 := 14\theta_2^5\theta_3^4\theta_4 + 2\theta_2\theta_3^8\theta_4 + 2\theta_2\theta_3^4\theta_4^5 + 16\theta_3^4\theta_4\theta_2'' - 16\theta_2\theta_3^4\theta_4'', \\ c_3 := 2\theta_2^5\theta_4^5 + 2\theta_2\theta_3^4\theta_4^5 + 2\theta_2\theta_4^9 + 16\theta_4^5\theta_2'' - 16\theta_2\theta_4^4\theta_4'', \\ c_4 := -12\theta_2^5\theta_3^2\theta_4^3. \end{gathered}$$

The reader may observe that if we write (1.1) and (1.2) as $\theta_2(0)^2\theta_2(z)^2 \equiv \theta_3(0)^2\theta_3(z)^2 - \theta_4(0)^2\theta_4(z)^2$ and $\theta_2(0)^2\theta_1(z)^2 \equiv \theta_3(0)^2\theta_4(z)^2 - \theta_4(0)^2\theta_3(z)^2$, and multiply them on both sides, we get

$$\tilde{c}_1\theta_3(z)^2\theta_4(z)^2 + \tilde{c}_2\theta_4(z)^4 + \tilde{c}_3\theta_3(z)^4 + \tilde{c}_4\theta_1(z)^2\theta_2(z)^2 \equiv 0, \tag{1.5}$$

a similar form as (1.4) but with $\tilde{c_1} := \theta_3^4 + \theta_4^4$, $\tilde{c}_2 = \tilde{c}_3 := -\theta_3^2\theta_4^2$ and $\tilde{c}_4 := -\theta_2^4$. One may wonder if (1.4) and (1.5) are the same identity. We will not discuss the connections between (1.4) and (1.5) in this paper. Again this question can be solved algorithmically. For more details we refer to Sections 6.1.1 and 6.3.2 of Ye (2016).

The framework used to solve Problem 1.1 is the theory of elliptic functions and modular forms. In particular, we have to use an essential tool, which is Algorithm 5.11 from Ye (2017). As a result, we provide Algorithm 3.9 for solving Problem 1.1.

However, we observed that in the literature most identities fitting into Problem 1.1 are also in a smaller class, in which the coefficient set $\mathbb{K}[\Theta]$ is replaced by a subalgebra

$$\mathbb{K}[\widetilde{\Theta}]_h := \{p(\theta_2(0), \theta_3(0), \theta_4(0)) : p \in \mathbb{K}[x,y,z] \text{ homogeneous}\}.$$

We define

$$R_2 := \mathbb{K}[\widetilde{\Theta}]_h[\theta_1(z|\tau), \theta_2(z|\tau), \theta_3(z|\tau), \theta_4(z|\tau)].$$

Restricting $\mathbb{K}[\Theta]$ to $\mathbb{K}[\widetilde{\Theta}]_h$, we provide Algorithm 6.6 to solve the following problem algorithmically without invoking Algorithm 5.11 of Ye (2017).

Problem 1.2. Given $f \in R_2$, decide whether $f = 0$.

[1] We use the notation $f_1(z_1, z_2, \ldots) \equiv f_2(z_1, z_2, \ldots)$ if we want to emphasize that the equality between the functions holds for all possible choices of the arguments z_j.

For example, Algorithm 6.6 can be used to prove identities like (1.1), (1.2), (1.3).

Algorithm 6.6 is faster than Algorithm 3.9 in our experiments. We will give some brief arguments concerning the speed comparison in the end of this paper. Moreover, working with this restricted class, we also found some classical mathematical insights, such as Proposition 5.2 and Lemma 5.8.

The paper is organized as follows. In Section 2 we present a theorem to decompose any $f(z|\tau) \in R_1$ into a set of quasi-elliptic components of $f(z|\tau)$, and prove that $f(z|\tau) \equiv 0$ if and only if its quasi-elliptic components are all equal to zero. In Section 3 we give an algorithm to decide if a quasi-elliptic component of any function in R_1 is equal to zero or not, thus we achieve the goal to prove or disprove $f(z|\tau) \equiv 0$. In Section 4 we derive a theorem connecting the Weierstrass elliptic function and the theta functions in a (new) way, which plays an important role for solving Problem 1.2. Working in the restricted space R_2, in Section 5 we obtain a critical lemma about the finite-orbit weight. In Section 6 we give an algorithm to decide whether any function in R_2 is equal to zero or not, thus we achieve the goal of solving Problem 1.2.

Convention. (i) Throughout the paper τ is always in the upper-half plane $\mathbb{H}$ and for $z = ce^{i\varphi}$ ($c > 0, 0 \leq \varphi < 2\pi$) we define $z^r := c^r e^{ir\varphi}$ for $r \in \frac{1}{2}\mathbb{Z}$.

(ii) For two sets A and B, we use B^A to present the set of functions $\{f\colon A \to B\}$.

(iii) For any $\alpha \in \mathbb{Z}^n$ we write $\alpha = (\alpha_1, \ldots, \alpha_n)$ and define $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

2. Decomposition into quasi-elliptic components

Definition 2.1. Given a finite set $M \subseteq \mathbb{N}^4$, define

$$f_M : \mathbb{K}[\Theta]^M \to R_1$$
$$\psi \mapsto f_M(\psi) =: f_M^\psi$$

where

$$f_M^\psi(z|\tau) := \sum_{\alpha \in M} \psi(\alpha)\theta^\alpha(z|\tau).$$

Notation. If M is clear from the context, we write f instead of f_M, and f^ψ instead of f_M^ψ. Sometimes, for convenience, we use $f^\psi(z)$ to present $f^\psi(z|\tau)$.

As an illustration of Definition 2.1, let us look at the identity (1.4). Here we have

$$M = \{(0,0,2,2), (0,0,0,4), (0,0,4,0), (2,2,0,0)\}, \tag{2.1}$$

$\psi((0,0,2,2)) = c_1$, $\psi((0,0,0,4)) = c_2$, $\psi((0,0,4,0)) = c_3$, $\psi((2,2,0,0)) = c_4$ and

$$f^\psi(z) = c_1\theta_3(z)^2\theta_4(z)^2 + c_2\theta_4(z)^4 + c_3\theta_3(z)^4 + c_4\theta_1(z)^2\theta_2(z)^2.$$

In order to decompose $f_M^\psi \in R_1$, we decompose the corresponding M first.

Definition 2.2. Given $\alpha, \beta \in \mathbb{N}^4$, we say that α, β are similar if

$$|\alpha| = |\beta|,\ \alpha_1 + \alpha_4 \equiv \beta_1 + \beta_4 (\operatorname{mod} 2) \text{ and } \alpha_1 + \alpha_2 \equiv \beta_1 + \beta_2 (\operatorname{mod} 2),$$

denoted by $\alpha \sim \beta$.

One can verify that $\sim$ is an equivalence relation.

Example 2.3. (i) $(0,0,2,2) \sim (0,0,0,4) \sim (0,0,4,0) \sim (2,2,0,0)$.
(ii) Let $M = \{(0,0,2,0),(0,0,0,2),(2,0,2,0),(2,1,1,0)\}$. Then

$$M/\sim = \{\{(0,0,2,0),(0,0,0,2)\},\{(2,0,2,0)\},\{(2,1,1,0)\}\}.$$

Definition 2.4. Let $f^\psi = f_M^\psi \in R_1$. If $M/\sim = \{M\}$, we say that f^ψ is quasi-elliptic.

Definition 2.5. Given $f^\psi = f_M^\psi \in R_1$ and $M/\sim = \{M_1,\ldots,M_n\}$, we define the set of quasi-elliptic components of f^ψ by

$$\{f_1,\ldots,f_n\},$$

where $f_j := f_{M_j}^{\psi_j}$ and $\psi_j := \psi|_{M_j}$.

Example 2.6. Let $M = \{(0,0,2,0),(0,0,0,2),(2,0,2,0),(2,1,1,0)\}$ and

$$f^\psi = f_M^\psi = c_1\theta_3(z)^2 + c_2\theta_4(z)^2 + c_3\theta_1(z)^2\theta_3(z)^2 + c_4\theta_1(z)^2\theta_2(z)\theta_3(z)$$

with the $c_j \in \mathbb{K}[\Theta]$. Then the set of quasi-elliptic components of f^ψ is

$$\{c_1\theta_3(z)^2 + c_2\theta_4(z)^2, c_3\theta_1(z)^2\theta_3(z)^2, c_4\theta_1(z)^2\theta_2(z)\theta_3(z)\}.$$

Theorem 2.7. Let $f^\psi = f_M^\psi \in R_1$ and $f_1,\ldots,f_n$ be the quasi-elliptic components of f^ψ, then

$$f^\psi(z|\tau) \equiv 0 \text{ if and only if } f_j(z|\tau) \equiv 0 \text{ for all } j \in \{1,\ldots,n\}.$$

Before we prove this theorem, we need to recall the following lemma.

Lemma 2.8. (Whittaker and Watson, 1927, p. 465) Let $N := e^{-\pi i\tau - 2iz}$. For $j \in \{1,2,3,4\}$ we have $\theta_j(z+\pi\tau|\tau) = \varepsilon_1(j)\theta_j(z|\tau)$ and $\theta_j(z+\pi|\tau) = \varepsilon_2(j)\theta_j(z|\tau)$ where $\varepsilon_1(j)$ and $\varepsilon_2(j)$ are defined in Table 2.1.

j	1	2	3	4
$\varepsilon_1(j)$	$-N$	N	N	$-N$
$\varepsilon_2(j)$	-1	-1	1	1

Table 2.1.

Proof of Theorem 2.7. If $f^{\psi_j}(z|\tau) \equiv 0$ for all $j \in \{1\ldots n\}$ then $f^\psi(z|\tau) \equiv 0$ is immediate.
Suppose $f^\psi(z|\tau) \equiv 0$. Write $f^\psi(z|\tau) := \sum_{\alpha\in M} \psi(\alpha)\theta^\alpha(z|\tau)$ and $\{d_1,\ldots,d_m\} := \{|\alpha| : \alpha \in M\}$.
Next we define

$$f_{t,0}(z) := \sum_{\substack{\alpha\in M \\ |\alpha|=d_t \\ \alpha_1+\alpha_4 \text{ even}}} \psi(\alpha)\theta^\alpha(z) \quad \text{and} \quad f_{t,1}(z) := \sum_{\substack{\alpha\in M \\ |\alpha|=d_t \\ \alpha_1+\alpha_4 \text{ odd}}} \psi(\alpha)\theta^\alpha(z).$$

By employing Table 2.1, we obtain for $t \in \{1,\ldots,m\}$,

$$f_{t,0}(z+\pi\tau) \equiv N^{d_t} f_{t,0}(z) \quad \text{and} \quad f_{t,1}(z+\pi\tau) \equiv -N^{d_t} f_{t,1}(z).$$

Then for $k \in \mathbb{Z}$,

$$f_{t,0}(z+k\pi\tau) + f_{t,1}(z+k\pi) \equiv (N^{d_t})^k f_{t,0}(z) + (-N^{d_t})^k f_{t,1}(z).$$

Thus the system of equations

$$0 \equiv f(z) \equiv f(z+k\pi\tau) \equiv \sum_{t=1}^{m} (N^{d_t})^k f_{t,0}(z) + (-N^{d_t})^k f_{t,1}(z) \quad (k \in \{0,1,\ldots,2m-1\})$$

can be written as

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ N^{d_1} & -N^{d_1} & \cdots & N^{d_m} & -N^{d_m} \\ (N^{d_1})^2 & (-N^{d_1})^2 & \cdots & (N^{d_m})^2 & (-N^{d_m})^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (N^{d_1})^{2m-1} & (-N^{d_1})^{2m-1} & \cdots & (N^{d_m})^{2m-1} & (-N^{d_m})^{2m-1} \end{pmatrix} \begin{pmatrix} f_{1,0} \\ f_{1,1} \\ \vdots \\ f_{m,0} \\ f_{m,1} \end{pmatrix} = 0. \tag{2.2}$$

Since $N \neq 0$, the determinant of this Vandermonde matrix is nonzero. Hence we obtain $f_{t,i} = 0$ for all $t \in \{1,\ldots,m\}$ and $i \in \{0,1\}$.

Next we write $f_{t,a} = f_{t,a,0} + f_{t,a,1}$ where

$$f_{t,0,0}(z) := \sum_{\substack{\alpha\in M \\ |\alpha|=d_t \\ \alpha_1+\alpha_4 \text{ even} \\ \alpha_1+\alpha_2 \text{ even}}} \psi(\alpha)\theta^{\alpha}(z) \quad \text{and} \quad f_{t,0,1}(z) := \sum_{\substack{\alpha\in M \\ |\alpha|=d_t \\ \alpha_1+\alpha_4 \text{ even} \\ \alpha_1+\alpha_4 \text{ odd}}} \psi(\alpha)\theta^{\alpha}(z),$$

$$f_{t,1,0}(z) := \sum_{\substack{\alpha\in M \\ |\alpha|=d_t \\ \alpha_1+\alpha_4 \text{ odd} \\ \alpha_1+\alpha_2 \text{ even}}} \psi(\alpha)\theta^{\alpha}(z) \quad \text{and} \quad f_{t,1,1}(z) := \sum_{\substack{\alpha\in M \\ |\alpha|=d_t \\ \alpha_1+\alpha_4 \text{ odd} \\ \alpha_1+\alpha_4 \text{ odd}}} \psi(\alpha)\theta^{\alpha}(z).$$

Again by using Table 2.1, we obtain for $a \in \{0,1\}$,

$$0 \equiv f_{t,a}(z+\pi) \equiv f_{t,a,0}(z+\pi) + f_{t,a,1}(z+\pi) \equiv f_{t,a,0}(z) - f_{t,a,1}(z).$$

This together with $0 = f_{t,a} = f_{t,a,0} + f_{t,a,1}$ implies $f_{t,a,0} = f_{t,a,1} = 0$ for all $t \in \{1,\ldots,m\}$ and $a \in \{0,1\}$.

In view of Definitions 2.2 and 2.5 the quasi-elliptic components of f^{Ψ} are exactly $f_{t,a,0}, f_{t,a,1}$ with $t \in \{1,\ldots,m\}$ and $a \in \{0,1\}$. $\square$

3. Zero-recognition for $f^{\Psi} \in R_1$

In this section we will use elliptic function properties to decide whether any given $f^{\Psi} \in R_1$ is identically zero.

Lemma 3.1. Given $\alpha, \beta \in \mathbb{N}^4$, if $\alpha \sim \beta$ then $\frac{\theta^{\alpha}(z|\tau)}{\theta^{\beta}(z|\tau)}$ is an elliptic function of z.

Proof. We have to show that the quotient is periodic in two directions. For this purpose, we apply Lemma 2.1:

$$\frac{\theta^{\alpha}(z+\pi\tau)}{\theta^{\beta}(z+\pi\tau)} \equiv \frac{(-1)^{\alpha_1+\alpha_4} N^{|\alpha|} \theta^{\alpha}(z)}{(-1)^{\beta_1+\beta_4} N^{|\beta|} \theta^{\beta}(z)} \equiv \frac{\theta^{\alpha}(z)}{\theta^{\beta}(z)},$$

where we exploited the fact that $\alpha \sim \beta$. Similarly we can show

$$\frac{\theta^\alpha(z+\tau)}{\theta^\beta(z+\tau)} \equiv \frac{(-1)^{\alpha_1+\alpha_2} N^{|\alpha|} \theta^\alpha(z)}{(-1)^{\beta_1+\beta_2} N^{|\beta|} \theta^\beta(z)} \equiv \frac{\theta^\alpha(z)}{\theta^\beta(z)}.$$

□

Definition 3.2. Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$. A period-parallelogram with periods ω_1 and ω_2 is denoted by

$$P(\omega_1, \omega_2) := \{t_1\omega_1 + t_2\omega_2 : t_1, t_2 \in [0,1[\}.$$

Note. In this paper, $\omega_1 = \pi$ and $\omega_2 = \pi\tau$. Usually in the literature, e.g. in (Chandrasekharan, 1985, p. 23, Th. 3), (Jones and Singerman, 2005, p. 75, Th. 3.6.4) and (Whittaker and Watson, 1927, p. 432), a different definition of parallelogram with periods ω_1 and ω_2 is given by

$$\bar{P}(\omega_1, \omega_2) := \{t_1\omega_1 + t_2\omega_2 : t_1, t_2 \in [0,1]\}.$$

Proposition 3.3. (Whittaker and Watson, 1927, 21.12) For each $j \in \{1,2,3,4\}$, $\theta_j(z)$ has only one zero in $P(\pi, \pi\tau)$. The zeros of $\theta_1(z)$, $\theta_2(z)$, $\theta_3(z)$, $\theta_4(z)$ are at the points congruent respectively to 0, $\frac{\pi}{2}$, $\frac{\pi}{2} + \frac{\pi\tau}{2}$, $\frac{\pi\tau}{2}$, modulo $\{m\pi + n\pi\tau : m, n \in \mathbb{Z}\}$.

Definition 3.4. Given a meromorphic function f on $\mathbb{C}$, we define

$$\text{poles}(f) := \{z \in \mathbb{C} \colon f \text{ has a pole at } z\}$$

and

$$\text{zeros}(f) := \{z \in \mathbb{C} \colon f \text{ has a zero at } z\}.$$

We recall the following classical Lemma.

Lemma 3.5. (Chandrasekharan, 1985, p. 23, Th. 3) For any nonzero elliptic function f with periods ω_1 and ω_2, one has

$$\#\left(\text{poles}(f) \cap \bar{P}(\omega_1, \omega_2)\right) = \#\left(\text{zeros}(f) \cap \bar{P}(\omega_1, \omega_2)\right).$$

Note. $\text{poles}(f) \cap \bar{P}(\omega_1, \omega_2)$ and $\text{zeros}(f) \cap \bar{P}(\omega_1, \omega_2)$ are finite sets. From Proposition 3.3 we learn that $\theta_1, \ldots, \theta_4$ have in total 9 zeros in $\bar{P}(\omega_1, \omega_2)$, while they have only 4 zeros in $P(\omega_1, \omega_2)$. For the simplicity of our algorithm, i.e., to compute as less as possible, we want to work with $P(\omega_1, \omega_2)$ instead of $\bar{P}(\omega_1, \omega_2)$. Therefore we need to prove the following theorem.

Theorem 3.6. For any nonzero elliptic function f with periods ω_1 and ω_2, one has

$$\#\left(\text{poles}(f) \cap P(\omega_1, \omega_2)\right) = \#\left(\text{zeros}(f) \cap P(\omega_1, \omega_2)\right).$$

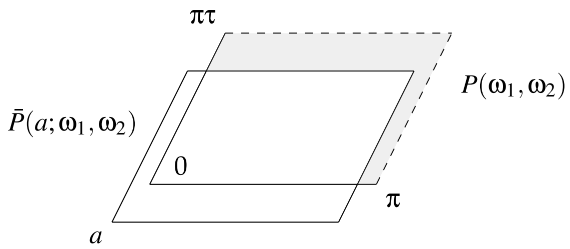
Proof. Let $H := \{z \in P(\omega_1, \omega_2) \colon f \text{ has a pole or zero at } z\}$, $h_1 := \max\{t_1 : t_1\omega_1 + t_2\omega_2 \in H \text{ with } t_1, t_2 \in [0,1[\}$ and $h_2 := \max\{t_2 : t_1\omega_1 + t_2\omega_2 \in H \text{ with } t_1, t_2 \in [0,1[\}$. We define a closed period parallelogram by

$$\bar{P}(a; \omega_1, \omega_2) := \{a + b\omega_1 + c\omega_2 : b, c \in [0,1]\}$$

with

$$a := -\frac{1-h_1}{2}\omega_1 - \frac{1-h_2}{2}\omega_2.$$

The following image visualizes the positions of $\bar{P}(a;\omega_1,\omega_2)$ and $P(\omega_1,\omega_2)$.



Let A denote

$$\{z : z \in P(\omega_1,\omega_2) \setminus \bar{P}(a;\omega_1,\omega_2)\}$$

plus the line segments where $P(\omega_1,\omega_2)$ intersects the boundary of $\bar{P}(a;\omega_1,\omega_2)$; let B denote

$$\{z : z \in \bar{P}(a;\omega_1,\omega_2) \setminus P(\omega_1,\omega_2)\}.$$

By the definition of $\bar{P}(a;\omega_1,\omega_2)$, one can easily check that (i) $f(z)$ has no poles or zeros in A; and (ii) for any $y \in P(\omega_1,\omega_2)$ if y is a zero (or a pole) of $f(z)$, then y is also in the interior of $\bar{P}(a;\omega_1,\omega_2)$.

Next, one can verify that (iii) for every $z \in \mathbb{C}$ there exists exactly one point $z_1 \in P(\omega_1,\omega_2)$ such that $z = z_1 + m\omega_1 + n\omega_2$ with $m,n \in \mathbb{Z}$; moreover, (iv) for ever point z in B the corresponding z_1 is in A.

On the other hand, (iii) yields that for any $z \in \mathbb{C}$ there exists exactly one point $z_1 \in P(\omega_1,\omega_2)$ such that $f(z) = f(z_1 + m\omega_1 + n\omega_2) = f(z_1)$.

Using (ii) and (iv), we deduce that $f(z)$ does not have any zeros or poles in B. Hence, $f(z)$ does not have any zeros or poles on the boundary of $\bar{P}(a;\omega_1,\omega_2)$, and the set of zeros and poles in $\bar{P}(a;\omega_1,\omega_2)$ are the same as the set of zeros and poles in $P(\omega_1,\omega_2)$. Applying Lemma 3.5 we complete the proof. □

Definition 3.7. Given a finite set $M \subseteq \mathbb{N}^4$, we define

$$\min(M,\psi) := \{(\beta_1,\beta_2,\beta_3,\beta_4) \in M : \beta_1 = \min\{\alpha_1 : \alpha \in M \text{ and } \psi(\alpha) \neq 0\}\}.$$

Lemma 3.8. Let $f^{\Psi}(z|\tau) := \sum_{\alpha\in M} \psi(\alpha)\theta^{\alpha}(z)$ be quasi-elliptic. For any $\beta = (\beta_1,\beta_2,\beta_3,\beta_4) \in \min(M,\psi)$, let $g_\beta(z|\tau) := \frac{f^{\Psi}(z|\tau)}{\theta^{\beta}(z)}$. Then $g_\beta(z|\tau)$ has a Taylor expansion

$$g_\beta(z|\tau) \equiv \sum_{j=0}^{\infty} d_j(\tau) z^j$$

with $d_j(\tau) \in \mathbb{K}(\Theta)$. [2]

Proof. From Definition 1.1 we know that for fixed $\tau \in \mathbb{H}$ the $\theta_j(z|\tau)$ ($j = 1,\ldots,4$) are analytic functions on the whole complex plane with respect to z, and for fixed $z \in \mathbb{C}$, the $\theta_j(z|\tau)$ ($j = 1,\ldots,4$) are analytic functions of τ for all $\tau \in \mathbb{H}$. By Proposition 3.3, only $\theta_1(z)$ has a zero at $z = 0$. Since all $\theta_1^{\beta_1}(z)$ in the denominator of $g_\beta(z|\tau)$ cancel against each $\theta^{\alpha}(z)$

[2] $\mathbb{K}(\Theta)$ denotes the quotient field of $\mathbb{K}[\Theta]$ consisting of all quotients $P(\Theta)/Q(\Theta)$ with $P(\Theta),Q(\Theta) \in \mathbb{K}[\Theta]$.

by the choice of β, we deduce that $g_\beta(z|\tau)$ is analytic at $z = 0$. Hence we have the Taylor expansion around $z = 0$.

□

Algorithm 3.9. Given $f^\Psi \in R_1$ with $f^\Psi = f_M^\Psi$, we have the following algorithm to decide whether $f^\Psi = 0$.

Input: $f^\Psi \in R_1$.
Output: True if $f^\Psi = 0$; False if $f^\Psi \neq 0$.

Compute the quasi-elliptic components of f^Ψ, denoted by $f_1, \ldots, f_n$ and $f_j = \sum_{\alpha \in M_j} \psi(\alpha)\theta^\alpha$.

for $j = 1, \ldots, n$ **do**
    take a random $\beta = (\beta_1, \beta_2, \beta_3, \beta_4) \in \min(M_j, \psi)$ and compute the expansion $\frac{f_j(z|\tau)}{\theta^\beta(z|\tau)} = \sum_{k=0}^{\infty} d_k(\tau) z^k$
    for $k = 0, \ldots, \beta_2 + \beta_3 + \beta_4$ **do**
        if $d_k(\tau) \equiv 0$ **then**
            $k \leftarrow k + 1$
        else return False
        end if
    end for
end for
return True

Note. In Algorithm 3.9, we use Algorithm 5.11 of Ye (2017) to check whether $d_k(\tau) \equiv 0$.

Theorem 3.10. Algorithm 3.9 is correct.

Proof. According to Definition 2.2 we can always write any $f^\Psi \in R_1$ into a sum of quasi-elliptic components of $f^\Psi = f_M^\Psi$ for some finite set $M \subseteq \mathbb{N}^4$. By Lemma 3.8 we always have a Taylor expansion of $\frac{f_j(z|\tau)}{\theta^\beta(z|\tau)}$ at $z = 0$.

Assume $f^\Psi = 0$. Then by Theorem 2.7, $f_j = 0$ for all $j \in \{1, \ldots, n\}$. Hence the corresponding $g = 0$, which implies $d_k(\tau) \equiv 0$ for all $k \in \mathbb{N}$. Therefore Algorithm 3.9 returns True.

Assume $f^\Psi \neq 0$. By Theorem 2.7, there exists a $t \in \{1, \ldots, n\}$ such that $f_t \neq 0$. Then the corresponding g is nonzero. If g is a constant, then $d_0 \neq 0$ and Algorithm 3.9 returns False. Assume g is not a constant. By Lemma 3.1, $g(z|\tau)$ is an elliptic function. Since $g(z|\tau)$ has at most $\beta_2 + \beta_3 + \beta_4$ poles in $P(\pi, \pi\tau)$, by Theorem 3.6 we deduce that $g(z|\tau)$ has at most $\beta_2 + \beta_3 + \beta_4$ zeros in $P(\pi, \pi\tau)$. This means $d_0(\tau), \ldots, d_{\beta_2+\beta_3+\beta_4}(\tau)$ cannot be all zero. Thus Algorithm 3.9 returns False. □

Example 3.11. Prove

$$f(z) := c_1\theta_3(z)^2\theta_4(z)^2 + c_2\theta_4(z)^4 + c_3\theta_3(z)^4 + c_4\theta_1(z)^2\theta_2(z)^2 \equiv 0,$$

where the c_j are chosen as in identity (1.4).

Proof. One can check by Definition 2.2 that f has only one quasi-elliptic component, itself. Following Algorithm 3.9,

$$g(z) := \frac{f(z)}{c_3\theta_4(z)^4} = c_1\frac{\theta_3(z)^2}{\theta_4(z)^2} + c_2 + c_3\frac{\theta_3(z)^4}{\theta_4(z)^4} + c_4\frac{\theta_1(z)^2\theta_2(z)^2}{\theta_4(z)^4}.$$

Then

$$g(z) = \sum_{k=0}^{\infty} d_k(\tau) z^k$$

with $d_0(\tau) = c_4\theta_1^2\theta_2^2 + c_3\theta_3^4 + c_1\theta_3^2\theta_4^2 + c_2\theta_4^4$ and $d_k(\tau)$ for $k = 1, \ldots, 4$ of a form similar to $d_0(\tau)$. By Algorithm 5.11 in Ye (2017) we can prove that $d_0 = \cdots = d_4 = 0$. Thus by Algorithm 3.9 we have $g = 0$. $\square$

Note. This identity contains only one quasi-elliptic component, and in general the identities we found in the literature are stated in their simplest form. Consequently, to produce an identity with more than one quasi-elliptic component, we need to take one identity containing one quasi-elliptic component (multiplied by an element of R_1) and add to it another identity containing one quasi-elliptic component (multiplied by an element of R_1).

The following proposition shows that there is a further decomposition step that can be done before doing zero-recognition.

Proposition 3.12. Given $f^{\Psi}(z) = \sum_{\alpha\in M} \psi(\alpha)\theta^{\alpha}(z) \in R_1$, then $f(z) \equiv 0$ if and only if

$$\sum_{\alpha\in N_i} \psi(\alpha)\theta^{\alpha}(z) \equiv 0$$

for $i = 1, 2$, where $N_1 := \{(\alpha_1, \ldots, \alpha_4) \in M : \alpha_1 \text{ is odd}\}$ and $N_2 := \{(\alpha_1, \ldots, \alpha_4) \in M : \alpha_1 \text{ is even}\}$.

Proof. If $\sum_{\alpha\in N_i} \psi(\alpha)\theta^{\alpha}(z) \equiv 0$ for $i = 1, 2$ then $f(z) \equiv 0$ is immediate. Assume $f(z) \equiv 0$. Let

$$f_i(z) := \sum_{\alpha\in N_i} \psi(\alpha)\theta^{\alpha}(z).$$

By Definition 1.1, $\theta_1(z)$ is an odd function while the other three are even functions, hence

$$0 \equiv f^{\Psi}(z) \equiv f^{\Psi}(-z) \equiv -f_1(z) + f_2(z).$$

This together with $f^{\Psi}(z) \equiv f_1(z) + f_2(z)$ implies $f_1(z) \equiv 0$ and $f_2(z) \equiv 0$. $\square$

4. Theta functions and Weierstrass $\wp$ function

We are going to derive some connections between theta functions and the $\wp$ function. By applying them, we will obtain a faster algorithm on the restricted class R_2.

Definition 4.1. [elliptic theta-quotients]

$$J := \{\theta^{\alpha}(z) : \alpha \in \mathbb{Z}^4 \text{ such that } \theta^{\alpha}(z) \text{ is elliptic}\}.$$

Lemma 4.2. *J* forms a multiplicative group which is generated by

$$j_1(z) := \frac{\theta_2(z)^2}{\theta_1(z)^2},\ j_2(z) := \frac{\theta_3(z)^2}{\theta_1(z)^2} \text{ and } j_3(z) := \frac{\theta_2(z)\theta_3(z)\theta_4(z)}{\theta_1(z)^3}.$$

In particular, for a given $p(z) = \theta_1(z)^{\alpha_1}\theta_2(z)^{\alpha_2}\theta_3(z)^{\alpha_3}\theta_4(z)^{\alpha_4} \in J$, the presentation in terms of the generators is

$$p = j_1^{\frac{\alpha_2-\alpha_4}{2}} j_2^{\frac{\alpha_3-\alpha_4}{2}} j_3^{\alpha_4}.$$

Proof. By the help of Table 2.1, one can verify that $j_1, j_2, j_3 \in J$ and that J is a multiplicative group. Suppose $p(z) = \theta_1(z)^{\alpha_1}\theta_2(z)^{\alpha_2}\theta_3(z)^{\alpha_3}\theta_4(z)^{\alpha_4}$, then $p(z) \in J$. Moreover, $p(z) \equiv p(z+\pi\tau)$ and $p(z) \equiv p(z+\pi)$, because every element in J is elliptic. On the other hand, by Table 2.1 we have

$$p(z+\pi\tau) \equiv (-1)^{\alpha_1+\alpha_4} N^{\alpha_1+\alpha_2+\alpha_3+\alpha_4} p(z) \text{ and } p(z+\pi) \equiv (-1)^{\alpha_1+\alpha_2} p(z).$$

Hence $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = 0$, $\alpha_1+\alpha_4$ is even and $\alpha_1+\alpha_2$ is even. This implies that if α_1 is even then also α_2, α_3 and α_4 must be even, and if α_1 is odd then also α_2, α_3 and α_4 must be odd. Therefore $\frac{\alpha_2-\alpha_4}{2}$, $\frac{\alpha_3-\alpha_4}{2}$ and α_4 are all integers. Moreover,

$$\begin{aligned} j_1^{\frac{\alpha_2-\alpha_4}{2}} j_2^{\frac{\alpha_3-\alpha_4}{2}} j_3^{\alpha_4} &= \theta_1^{-\alpha_2+\alpha_4-\alpha_3+\alpha_4-3\alpha_4}\theta_2^{\alpha_2-\alpha_4+\alpha_4}\theta_3^{\alpha_3-\alpha_4+\alpha_4}\theta_4^{\alpha_4} \\ &= \theta_1^{-\alpha_2-\alpha_3-\alpha_4}\theta_2^{\alpha_2}\theta_3^{\alpha_3}\theta_4^{\alpha_4} \\ &= \theta_1^{\alpha_1}\theta_2^{\alpha_2}\theta_3^{\alpha_3}\theta_4^{\alpha_4} \\ &= p. \end{aligned}$$

□

Proposition 4.3. (Freitag and Busam, 2005, p. 266, Prop. V.2.11) The Weierstrass $\wp$ function has a Laurent expansion

$$\wp(z;\omega_1,\omega_2) \equiv \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)E_{2k+2}(\omega_1,\omega_2)z^{2k},$$

where $E_{2k+2}(\omega_1,\omega_2) := \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m\omega_1+n\omega_2)^{-2k-2}$ is an Eisenstein series.

Theorem 4.4. The generators j_1, j_2 and j_3 of J satisfy

$$\left.\begin{aligned} j_1(z) &\equiv \tfrac{\theta_2(0)^2}{\theta_1(0)^2}(\wp(z)-e_1); \\ j_2(z) &\equiv \tfrac{\theta_3(0)^2}{\theta_1(0)^2}(\wp(z)-e_3); \end{aligned}\right\}\ {}^{3}$$
$$j_3(z) \equiv -\frac{1}{2\theta_1'(0)^2}\,\wp'(z),$$

where $\wp(z) := \wp(z,\pi,\pi\tau)$ is the Weierstrass elliptic function with periods π and $\pi\tau$, $e_1 := \frac{1}{3}(\theta_3(0)^4+\theta_4(0)^4)$ and $e_3 := \frac{1}{3}(\theta_2(0)^4-\theta_4(0)^4)$.

[3] See p. 102 of Köcher and Krieg (1985).

Proof. Recall the notation $\theta_j^{(k)} := \theta_j^{(k)}(0|\tau)$. Since $\frac{\theta_2(z)^2}{\theta_1(z)^2}$ is elliptic with a double pole at $z=0$ and is an even function, we can expand it as

$$\frac{\theta_2(z)^2}{\theta_1(z)^2} \equiv \frac{1}{z^2}\left(\frac{\theta_2^2}{\theta_1'^2} + \left(-\frac{\theta_2^2\theta_1^{(3)}}{3\theta_1'^3} + \frac{\theta_2\theta_2''}{\theta_1'^2}\right)z^2 + \cdots\right)$$
$$\equiv \frac{\theta_2^2}{\theta_1'^2}\frac{1}{z^2} + \left(-\frac{\theta_2^2\theta_1^{(3)}}{3\theta_1'^3} + \frac{\theta_2\theta_2''}{\theta_1'^2}\right) + \cdots .$$

Following Proposition 4.3,

$$\wp(z) \equiv \frac{1}{z^2} + \sum_{n=1}^{\infty} c_n z^{2n},$$

where $c_n = (2n+1)E_{2n+2}(\pi, \pi\tau)$. Thus $\wp(z) - \frac{\theta_1'^2}{\theta_2^2}\frac{\theta_2(z)^2}{\theta_1(z)^2}$ has no pole, which implies that it has to be constant, i.e.,

$$\wp(z) - \frac{\theta_1'^2}{\theta_2^2}\frac{\theta_2(z)^2}{\theta_1(z)^2} \equiv \frac{\theta_1^{(3)}}{3\theta_1'} - \frac{\theta_2''}{\theta_2} = \frac{1}{3}(\theta_3^4 + \theta_4^4) = e_1,$$

where the second last equality is proven using Algorithm 5.11 in Ye (2017). Thus

$$\frac{\theta_2(z)^2}{\theta_1(z)^2} \equiv \frac{\theta_2^2}{\theta_1'^2}(\wp(z) - e_1). \tag{4.1}$$

Analogously, we have

$$\wp(z) - \frac{\theta_1'^2}{\theta_3^2}\frac{\theta_3(z)^2}{\theta_1(z)^2} \equiv \frac{\theta_1^{(3)}}{3\theta_1'} - \frac{\theta_3''}{\theta_3} = \frac{1}{3}(\theta_2^4 - \theta_4^4) = e_3,$$

where the second last equality is proven using Algorithm 5.11 of Ye (2017), and thus

$$\frac{\theta_3(z)^2}{\theta_1(z)^2} \equiv \frac{\theta_3^2}{\theta_1'^2}(\wp(z) - e_3). \tag{4.2}$$

One can verify that $j_3(z) = \frac{\theta_2(z)\theta_3(z)\theta_4(z)}{\theta_1(z)^3} \in J$ is an odd elliptic function, and we have the series expansion

$$\frac{\theta_2(z)\theta_3(z)\theta_4(z)}{\theta_1(z)^3} \equiv a_{-3}z^{-3} + a_{-1}z^{-1} + a_1 z + \cdots,$$

where

$$a_{-3} := \frac{\theta_2\theta_3\theta_4}{\theta_1'^3},$$

$$a_{-1} = \frac{1}{2\theta_1'^5}(\theta_3\theta_4\theta_1'^2\theta_2'' + \theta_2\theta_4\theta_1'^2\theta_3'' + \theta_2\theta_3\theta_1'^2\theta_4'' + \theta_2\theta_3\theta_1'\theta_1^{(3)}),$$

and a_1 is also in $\mathbb{K}(\Theta)$ but irrelevant to this proof. We have checked with Algorithm 5.11 of Ye (2017) that a_{-1} is zero. From Proposition 4.3 we derive

$$\wp'(z) \equiv -\frac{2}{z^3} + \sum_{k=1}^{\infty} 2k(2k+1)E_{2k+2}z^{2k-1}. \tag{4.3}$$

Therefore

$$\frac{\theta_2(z)\theta_3(z)\theta_4(z)}{\theta_1(z)^3} + \frac{1}{2}\frac{\theta_2\theta_3\theta_4}{\theta_1'^3}\wp'(z) \tag{4.4}$$

has no poles and therefore has to be constant. We take $z=0$ and it turns out that the expression (4.4) is equal to zero. Thus

$$\frac{\theta_2(z)\theta_3(z)\theta_4(z)}{\theta_1(z)^3} \equiv -\frac{1}{2}\frac{\theta_2\theta_3\theta_4}{{\theta'_1}^3}\wp'(z) \equiv -\frac{1}{2{\theta'_1}^2}\wp'(z),$$

where the last equality follows from the famous identity

$$\theta'_1 = \theta_2\theta_3\theta_4,$$

which can be also proven by Algorithm 5.11 of Ye (2017). $\square$

Remark. Replacing z by $\frac{\pi}{2}$ in (4.1) and using $\theta_2\left(\frac{\pi}{2}\right)=0$, we obtain $\wp\left(\frac{\pi}{2}\right)=e_1$; substituting z by $\frac{\pi+\pi\tau}{2}$ in (4.2) and using $\theta_3\left(\frac{\pi+\pi\tau}{2}\right)=0$ gives $\wp\left(\frac{\pi+\pi\tau}{2}\right)=e_3$. It can be verified that $\frac{\theta_3(z)^2}{\theta_1(z)^2}$ is also elliptic, and similarly we have

$$\frac{\theta_4(z)^2}{\theta_1(z)^2} \equiv \frac{\theta_4^2}{{\theta'_1}^2}(\wp(z)-e_2)$$

where $e_2 := -\frac{1}{3}(\theta_2^4+\theta_3^4)$. Moreover, by $\theta_4(\frac{\pi\tau}{2})=0$ we obtain $\wp\left(\frac{\pi\tau}{2}\right)=e_2$.

5. The finite-orbit weight

In this section we will show the particularity of R_2, in terms of the finite-orbit weight, which will be used in the next section as a crucial property.

Definition 5.1. Let $M(\mathbb{H}) := \{g\colon g \text{ meromorphic on } \mathbb{H}\}$. Given $k\in\mathbb{Z}$, we define a group action

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Z})\times M(\mathbb{H}) &\longrightarrow M(\mathbb{H}) \\ (\rho, g) &\mapsto g|_k\rho \end{aligned}$$

where $g|_k\rho(\tau) := (c\tau+d)^{-k}g\left(\frac{a\tau+b}{c\tau+d}\right)$ for $\rho := \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})$ and $\tau\in\mathbb{H}$. For each $k\in\mathbb{Z}$ we define the k-orbit of g by $G_k(g) := \{g|_k\rho\colon \rho\in\mathrm{SL}_2(\mathbb{Z})\}$.

Proposition 5.2. For a nonzero $g\in M(\mathbb{H})$, there exists at most one $k\in\mathbb{Z}$ such that $|G_k(g)|$ is finite.

Proof. Let k and t be integers such that $G_k(g)$ and $G_t(g)$ are both finite orbit sets. We need to prove that $k=t$. Let $s := k-t$. Take any $g|_t\rho\in G_t(g)$ with $\rho = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$. Then

$$\begin{aligned} g|_t\rho(\tau) &= (c\tau+d)^{-t}g\left(\frac{a\tau+b}{c\tau+d}\right) \\ &= (c\tau+d)^{s}(c\tau+d)^{-k}g\left(\frac{a\tau+b}{c\tau+d}\right) \\ &= (c\tau+d)^{s}\cdot g|_k\rho(\tau). \end{aligned}$$

Hence we can rewrite the set $G_t(g)$ as

$$G_t(g) = \left\{ (c\tau+d)^s \cdot g|_k\rho \colon \rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\}$$
$$= \left\{ (c\tau+d)^s \cdot g_{a,b,c,d} \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } g_{a,b,c,d} \in G_k(g) \right\},$$

where $g_{a,b,c,d} := g|_k\rho$ with $\rho = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Assume $s \neq 0$ and $G_k(g) = \{a_1, \ldots, a_n\}$, and define the map

$$\begin{aligned} \phi \colon \mathrm{SL}_2(\mathbb{Z}) &\to G_k(g) \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) &\mapsto g_{a,b,c,d}. \end{aligned}$$

Let $A_j := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \colon g_{a,b,c,d} = a_j \right\}$. By Definition 5.1, the map ϕ is surjective, thus $A_j \neq \emptyset$. Then we can write $\mathrm{SL}_2(\mathbb{Z}) = \bigcup\limits_{j=1}^{n} A_j$ where $A_i \cap A_j = \emptyset$ if $i \neq j$. Let

$$B_j := \left\{ (c,d) \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A_j \right\}.$$

For every pair $(c,d) \in \mathbb{Z}^2$ with $\gcd(c,d) = 1$, there must exist some pairs $(a,b) \in \mathbb{Z}^2$ such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Hence there exists $r \in \{1, \ldots, n\}$ such that B_r is infinite; otherwise $\mathrm{SL}_2(\mathbb{Z}) \neq \bigcup\limits_{j=1}^{n} A_j$. We also have

$$\left\{ (c\tau+d)^s a_r \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A_r \right\} \subseteq \left\{ (c\tau+d)^s g_{a,b,c,d} \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\} = G_t(g),$$

which implies

$$N := \left| \left\{ (c\tau+d)^s a_r \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A_r \right\} \right| \leq |G_t(g)|. \tag{5.1}$$

On the other hand

$$N = |\{(c\tau+d)^s \colon (c,d) \in B_r\}|, \tag{5.2}$$

and the right hand side of (5.2) is equal to infinity because $c_1\tau + d_1 \neq c_2\tau + d_2$ when $(c_1,d_1) \neq (c_2,d_2)$, and because the set B_r is infinite. Thus N is equal to infinity, and by (5.1), $|G_t(g)| = \infty$, which contradicts the assumption that $G_t(g)$ is a finite orbit set. Therefore $s = 0$. $\square$

Definition 5.3. Let $g \in M(\mathbb{H})$ be nonzero and $k \in \mathbb{Z}$ be the unique number such that $|G_k(g)|$ is finite, we define the finite-orbit weight of g by

$$W(g) := k.$$

By using Definitions 5.1 and 5.3 one can verify the following:

Proposition 5.4. Given $g_1, \ldots, g_n \in M(\mathbb{H})$ with $W(g_j) = k_j$. Then
(1) $W(g_1 \cdots g_n) = k_1 + \cdots + k_n$,
(2) If $k_1 = \cdots = k_n = k$ and $g_1 + \cdots + g_n \neq 0$, then $W(g_1 + \cdots + g_n) = k$.

Lemma 5.5. (Serre, 1973, p. 78, Thm. 2) The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Note. According to this lemma, $\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \rangle$, hence

$$G_k(g) = \{g|_k\rho \colon \rho \in \langle S, T \rangle\}.$$

Thus in our working frame, to compute $G_k(g)$, we compute $\{g|_k\rho \colon \rho \in \langle S, T \rangle\}\}$.

Lemma 5.6. (Whittaker and Watson, 1927, p. 475) For the action of S on $\theta_j(z|\tau)$ ($j = 1, \ldots, 4$) we have

$$\theta_1\left(z\middle| -\frac{1}{\tau}\right) \equiv -i(-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_1(z\tau|\tau); \quad \theta_2\left(z\middle| -\frac{1}{\tau}\right) \equiv (-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_4(z\tau|\tau);$$
$$\theta_3\left(z\middle| -\frac{1}{\tau}\right) \equiv (-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_3(z\tau|\tau); \quad \theta_4\left(z\middle| -\frac{1}{\tau}\right) \equiv (-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_2(z\tau|\tau).$$

Directly from Definition 1.1 one can deduce the following.

Lemma 5.7. For the action of T on $\theta_j(\tau)$ ($j = 1, \ldots, 4$) we have

$$\theta_1(z|\tau+1) \equiv e^{\frac{\pi i}{4}} \theta_1(z|\tau); \quad \theta_2(z|\tau+1) \equiv e^{\frac{\pi i}{4}} \theta_2(z|\tau);$$
$$\theta_3(z|\tau+1) \equiv \theta_4(z|\tau); \quad \theta_4(z|\tau+1) \equiv \theta_3(z|\tau).$$

Now we show the special property of functions in R_2.

Lemma 5.8. Let $f^{\Psi}(z|\tau) = \sum_{\alpha \in M} \psi(\alpha)\theta^{\alpha}(z) \in R_2$ be quasi-elliptic and $\beta \in \min(M, \psi)$. Suppose the series expansion of $\frac{f^{\Psi}(z|\tau)}{\psi(\beta)\theta^{\beta}(z)}$ around $z = 0$ is of the form $\sum_{n=0}^{\infty} d_n(\tau) z^n$ with $d_n(\tau) \in \mathbb{K}(\Theta)$. Then $W(d_n) = n$ when $d_n \neq 0$.

Proof. By Lemma 3.8, $\frac{f^{\Psi}(z|\tau)}{\psi(\beta)\theta^{\beta}(z)}$ has a Taylor expansion around $z = 0$. Since $f^{\Psi}(z|\tau)$ is quasi-elliptic, by Theorem 3.1, $\frac{\theta^{\alpha}(z)}{\theta^{\beta}(z)}$ is elliptic for every $\alpha \in M$.

In view of $\frac{f^{\Psi}(z|\tau)}{\psi(\beta)\theta^{\beta}(z)} = \sum_{\alpha \in M} \frac{\psi(\alpha)\theta^{\alpha}(z)}{\psi(\beta)\theta^{\beta}(z)}$, we are going to show that the assertion is true for every $\frac{\psi(\alpha)\theta^{\alpha}(z)}{\psi(\beta)\theta^{\beta}(z)}$, then we show the assertion is true for $\frac{f^{\Psi}(z|\tau)}{\psi(\beta)\theta^{\beta}(z)}$. For any fixed $\alpha \in M$, by Lemma 4.2 and Theorem 4.4 there exist integers a, b, c, such that

$$\frac{\theta^{\alpha}(z)}{\theta^{\beta}(z)} \equiv \left(-\frac{1}{2}\right)^c \cdot \frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}} p(z) \tag{5.3}$$

where $p(z) := (\wp(z) - e_1)^a (\wp(z) - e_3)^b \wp'(z)^c$.

Applying Lemmas 5.6 and 5.7 one can verify that $W(\theta_2(0)^2) = 1$ and

$$G_1(\theta_2(0)^2) = \{\pm\theta_2(0)^2, \pm i\theta_2(0)^2, \pm\theta_3(0)^2, \pm i\theta_3(0)^2, \pm\theta_4(0)^2, \pm i\theta_4(0)^2\}.$$

Similarly we have $W(\theta_3(0)^2) = 1$ and $W(\theta_1'(0)^2) = 3$. Then by Proposition 5.4 (1) we obtain

$$\begin{aligned} W\left(\frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}}\right) &= W(\theta_2(0)^{2a}\theta_3(0)^{2b}) - W(\theta_1'(0)^{2a+2b+2c}) \\ &= W(\theta_2(0)^{2a}) + W(\theta_3(0)^{2b}) - W(\theta_1'(0)^{2a+2b+2c}) \\ &= a + b - 3a - 3b - 3c \\ &= -2a - 2b - 3c. \end{aligned}$$

Next we compute $W([z^n]p(z))$, where by $[z^n]p(z)$ we mean the coefficient of z^n in the series expansion of $p(z)$ around $z = 0$. Let us first consider

$$p_1(z) := z^{2a+2b+3c}p(z) = z^{2a}(\wp(z) - e_1)^a z^{2b}(\wp(z) - e_3)^b z^{3c}\,\wp'(z)^c. \tag{5.4}$$

Let $g_1(z) := z^2(\wp - e_1)$. By Proposition 4.3 we have

$$g_1(z) \equiv 1 - e_1 z^2 + \sum_{m=1}^{\infty}(2m+1)E_{2m+2}z^{2m+2}$$

where $E_{2m+2} := \sum_{\omega\in L,\omega\neq 0} \omega^{-(2m+2)}$ is an Eisenstein series and L is the lattice generated by π and $\pi\tau$. One can easily verify by using Definition 5.1 that $W(1) = 0$. Again using Lemma 5.6 and Lemma 5.7 one can verify that $W(e_1) = 2$. In addition, according to (Serre, 1973, p. 83) for $m \geq 1$,

$$W(E_{2m+2}) = W\left(\sum_{\omega\in L,\omega\neq 0} \omega^{-(2m+2)}\right) = 2m + 2.$$

Therefore, for any $n \geq 0$, if $[z^n]g_1(z) \neq 0$ then

$$W([z^n]g_1(z)) = n. \tag{5.5}$$

Next we do a case distinction on the power of $g_1(z)$ in (5.4).

Case 1: $a \geq 0$. Then

$$W\left([z^n]g_1(z)^a\right) = W\left(\sum_{n_1+n_2+\cdots+n_a=n} [z^{n_1}]g_1(z)\cdots[z^{n_a}]g_1(z)\right).$$

By (5.5) and by Proposition 5.4 (1), for any combination $n_1, \ldots, n_a$ such that $n_1 + \cdots + n_a = n$ we have

$$\begin{aligned} W([z^{n_1}]g_1(z)\cdots[z^{n_a}]g_1(z)) &= W([z^{n_1}]g_1(z)) + \cdots + W([z^{n_a}]g_1(z)) \\ &= n_1 + \cdots + n_a \\ &= n. \end{aligned}$$

Hence if $a \geq 0$, we find that

$$W\left([z^n]g_1(z)^a\right) = n \quad \text{when } [z^n]g_1(z)^a \neq 0.$$

Case 2: $a < 0$. Then

$$W\left([z^n]g_1(z)^a\right) = W\left([z^n]\left(\frac{1}{g_1(z)}\right)^{-a}\right)$$
$$= W\left(\sum_{n_1+n_2+\cdots+n_{-a}=n}[z^{n_1}]\left(\frac{1}{g_1(z)}\right)\cdots[z^{n-a}]\left(\frac{1}{g_1(z)}\right)\right).$$

Assuming $g_1(z) = \sum_{j=0}^{\infty} v_j z^j$ we have $\frac{1}{g_1(z)} = \sum_{j=0}^{\infty} u_j z^j$, noting that $v_0 = u_0 = 1$. We have proven that for all $n \geq 0$, $W(v_n) = n$ when $v_n \neq 0$. Now we prove that $W(u_n) = n$ when $u_n \neq 0$ by induction on n. When $n = 0$ we have $W(u_0) = W(v_0) = 0$. Assume for $n \leq N$, $W(u_n) = n$. Let $n = N+1$. Using $\sum_{j=0}^{\infty} v_j z^j \cdot \sum_{j=0}^{\infty} u_j z^j = 1$ we obtain

$$u_{N+1} = -\frac{v_1 u_N + v_2 u_{N-1} + \cdots + v_N u_1 + v_{N+1} u_0}{v_0} = -v_1 u_N - v_2 u_{N-1} - \cdots - v_N u_1 - v_{N+1}.$$

By Proposition 5.4 (2), if $u_{N+1} \neq 0$, then

$$W(u_{N+1}) = W(-v_1 u_N - v_2 u_{N-1} - \cdots - v_N u_1 - v_{N+1}) = N+1. \tag{5.6}$$

Hence $W(u_n) = n$ when $u_n \neq 0$. For any combination $n_1, \ldots, n_{-a}$ such that $n_1 + \cdots + n_{-a} = n$ we have

$$W\left([z^{n_1}]\left(\frac{1}{g_1(z)}\right)\cdots[z^{n_a}]\left(\frac{1}{g_1(z)}\right)\right) = W\left([z^{n_1}]\left(\frac{1}{g_1(z)}\right)\right) + \cdots + W\left([z^{n_a}]\left(\frac{1}{g_1(z)}\right)\right)$$
$$= n_1 + \cdots + n_{-a}$$
$$= n.$$

Again by Proposition 5.4.2 and by (5.6), for any $a < 0$ we find that

$$W\left([z^n]g_1(z)^a\right) = n \quad \text{when } [z^n]g_1(z)^a \neq 0.$$

Analogously we deduce that for $b, c \in \mathbb{Z}$,

$$W\left([z^n]z^{2b}(\wp - e_3)^b\right) = n \quad \text{and} \quad W([z^n]z^{3c}\wp'(z)^c) = n$$

whenever the function to which W is applied is nonzero. Consequently we deduce that when $[z^n]p_1(z) \neq 0$,

$$W\left([z^n]p_1(z)\right) = W\left([z^n]z^{2a}(\wp(z) - e_1)^a z^{2b}(\wp(z) - e_3)^b z^{3c}\wp'(z)^c\right)$$
$$= W\left(\sum_{n_1+n_2+n_3=n}[z^{n_1}](\wp(z) - e_1)^a \cdot [z^{n_1}]z^{2b}(\wp(z) - e_3)^b \cdot [z^{n_3}]z^{3c}\wp'(z)^c\right)$$
$$= n_1 + n_2 + n_3$$
$$= n,$$

where the second last equality follows from Proposition 5.4.1. This implies when $[z^n]p(z) \neq 0$,

$$W\left([z^n]p(z)\right) = W\left([z^{n+2a+2b+3c}]p_1(z)\right) = n + 2a + 2b + 3c. \tag{5.7}$$

Therefore if $[z^n]\frac{\theta^\alpha(z)}{\theta^\beta(z)} \neq 0$, identity (5.3) implies

$$\begin{aligned} W\left([z^n]\frac{\theta^\alpha(z)}{\theta^\beta(z)}\right) &= W\left((-\frac{1}{2})^c \cdot \frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}}p(z)\right) \\ &= W\left(\frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}}\right) + W\left([z^{2n}]p(z)\right) \\ &= -2a-2b-3c+n+2a+2b+3c \\ &= n. \end{aligned}$$

Moreover, since both $\psi(\alpha)$ and $\psi(\beta)$, by definition of β, are homogeneous polynomials in $\mathbb{K}[\widetilde{\Theta}]_h$ with the same degree, one can check, by using Lemmas 5.6 and 5.7, that $W\left(\frac{\psi(\alpha)}{\psi(\beta)}\right) = 0$ for all $\alpha \in M$. Hence

$$W\left([z^n]\frac{\psi(\alpha)\theta^\alpha(z)}{\psi(\beta)\theta^\beta(z)}\right) = 0+n = n \text{ when } [z^n]\frac{\psi(\alpha)\theta^\alpha(z)}{\psi(\beta)\theta^\beta(z)} \neq 0$$

and

$$W(d_n) = \sum_{\alpha\in M}[z^n]\frac{\psi(\alpha)\theta^\alpha(z)}{\psi(\beta)\theta^\beta(z)} = n \text{ when } d_n \neq 0.$$

□

6. Zero-recognition for $f^\Psi \in R_2$

Let us recall Definition 4.1. By Lemma 4.2 and Theorem 4.4, for any $\frac{\theta^\alpha}{\theta^\beta} \in J$ with $\alpha = (\alpha_1,\ldots,\alpha_4)$ and $\beta = (\beta_1,\ldots,\beta_4)$, we can write

$$\frac{\theta^\alpha(z)}{\theta^\beta(z)} \equiv \left(-\frac{1}{2}\right)^c \cdot \frac{\theta_2(0)^{2a}\theta_3(0)^{2b}}{\theta_1'(0)^{2a+2b+2c}}p(z), \tag{6.1}$$

where $p(z) := (\wp(z)-e_1)^a(\wp(z)-e_3)^b\,\wp'(z)^c$. The function $p(z)$ has the following property.

Proposition 6.1. Let $p(z)$ be the same as above and let g_n denote the coefficient of z^n in the series expansion of $p(z)$ around $z=0$. Then when $g_n \neq 0$ we have

$$|G_{w_n}(g_n)| \leq 3 = |G_2(e_1)|,$$

where w_n is the finite-orbit weight of g_n.

To prove this proposition, we need introduce the following definition.

Definition 6.2. (Freitag and Busam, 2005, p. 326) Let $q = e^{\pi i\tau}$ with $\tau \in \mathbb{H}$. Given $k \in \mathbb{N}$, a modular form of weight k is an analytic function g on $\mathbb{H}$ such that

$$g\left(\frac{a\tau+b}{c\tau+d}\right) \equiv (c\tau+d)^k g(\tau) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

and $g(\tau)$ can be written as a Taylor series in powers of q with complex coefficients; i.e.,

$$g(\tau) \equiv \sum_{t=0}^{\infty} a_t e^{\pi i\tau j} \equiv \sum_{j=0}^{\infty} a_j q^j.$$

Proof of Proposition 6.1. From the proof of Lemma 5.8 we observe that g_n is a polynomial in e_1, e_3 and E_{2s+2} with some $s \geq 1$. Also, according to equation (5.7),

$$W(g_n) = n + 2a + 2b + 3c =: n'$$

when $g_n \neq 0$. Let $p_1, \dots, p_t$ be the monomials of g_n, where each such monomial is a (finite) power product $e_1^{k_1} e_2^{k_2} E_4^{\ell_1} E_6^{\ell_2} \cdots$ with a coefficient in $\mathbb{K}$. One has

$$|G_{n'}(g_n)| = |\{(p_1 + \dots + p_t)|_{n'}\rho \colon \rho \in \mathrm{SL}_2(\mathbb{Z})\}|. \tag{6.2}$$

Additionally, from the proof of Lemma 5.8, the p_i in (6.2) are of the form

$$e_1^{k_1} e_3^{k_2} \prod_{s \in M_i} E_{2s+2}^{\ell_s}$$

where $k_1, k_2, \ell_s \in \mathbb{N}$, $M_i \subseteq \mathbb{N}$ and $2k_1 + 2k_2 + \sum_{s \in M_i} (2s+2)\ell_s = n'$. It can be verified by using Lemma 5.6 and Lemma 5.7 that $G_2(e_1) = G_2(e_3) = \{e_1, e_2, e_3\}$, thus $W(e_1) = W(e_3) = 3$.

By (Serre, 1973, p. 83) we have that if $s \geq 1$ then E_{2s+2} is a modular form of weight $2s+2$, which means

$$E_{2s+2}|_{2s+2}\rho = E_{2s+2} \text{ for all } \rho \in \mathrm{SL}_2(\mathbb{Z}).$$

Consequently,

$$E_{2s+2}^{\ell_s}|_{(2s+2)\ell_s}\rho = E_{2s+2}^{\ell_s} \quad \text{for all } \rho \in \mathrm{SL}_2(\mathbb{Z}).$$

By Proposition 5.4 (1),

$$W\left(\prod_{s \in M_i} E_{2s+2}^{\ell_s}\right) = \sum_{s \in M_i} (2s+2)\ell_s.$$

By Proposition 5.4 (2) we obtain $W(p_i) = 2k_1 + 2k_2 + \sum_{s \in M_i} (2s+2)\ell_s = n'$ for all $i \in \{1, \dots, t\}$. Hence we continue (6.2) by

$$\begin{aligned}
|G_{n'}(g_n)| &= \{(p_1 + \dots + p_t)|_{n'}\rho : \rho \in \mathrm{SL}_2(\mathbb{Z})\}| \\
&\leq |\{\{p_1|_{n'}\rho, \dots, p_t|_{n'}\rho\} \colon \rho \in \mathrm{SL}_2(\mathbb{Z})\}| \\
&= |\{\{(e_1^{k_{1,1}} e_3^{k_{1,2}} \gamma_1)|_{n'}\rho, \dots, (e_1^{k_{t,1}} e_3^{k_{t,2}} \gamma_t)|_{n'}\rho\} \colon \rho \in \mathrm{SL}_2(\mathbb{Z})\}| \\
&= |\{\{(e_1^{k_{1,1}} e_3^{k_{1,2}})|_{2(k_{1,1}+k_{1,2})}\rho, \dots, (e_1^{k_{t,1}} e_3^{k_{t,2}})|_{2(k_{1,1}+k_{1,2})}\rho\} \colon \rho \in \mathrm{SL}_2(\mathbb{Z})\}|,
\end{aligned} \tag{6.3}$$

where the γ_i are the corresponding $\prod_{m \in M_i} E_{2s+2}^{\ell_s}$ of p_i. On the other hand, for $k \in \mathbb{N}$,

$$G_{2k}(e_1^k) = \{e_1^k|_{2k}\rho : \rho \in \mathrm{SL}_2(\mathbb{Z})\} = \{\underbrace{e_1|_2\rho \cdots e_1|_2\rho}_{k} \colon \rho \in \mathrm{SL}_2(\mathbb{Z})\} = \{e_1^k, e_2^k, e_3^k\},$$

and analogously $G_{2k}(e_2^k) = G_{2k}(e_3^k) = \{e_1^k, e_2^k, e_3^k\}$. Then

$$\{e_1^{k_1} e_3^{k_2}|_{2(k_1+k_2)}\rho : \rho \in \mathrm{SL}_2(\mathbb{Z})\} = \{e_1^{k_1}|_{2k_1}\rho \cdot e_2^{k_2}|_{2k_2}\rho : \rho \in \mathrm{SL}_2(\mathbb{Z})\} = \{e_2^{k_1} e_1^{k_2}, e_3^{k_1} e_2^{k_2}, e_1^{k_1} e_3^{k_2}\},$$

which means there are only three possibilities when applying an arbitrary $\rho \in \mathrm{SL}_2(\mathbb{Z})$ on every $e_1^{k_{i,1}} e_3^{k_{i,2}}$ of (6.3). Note that the powers $k_{i,j}$ are irrelevant, i.e., we can choose three representatives ρ_1, ρ_2 and ρ_3 such that for all $i \in \{1, \dots, t\}$,

$$(e_1^{k_{i,1}} e_3^{k_{i,2}})|_{2(k_{i,1}+k_{i,2})}\rho_1 = e_2^{k_{i,1}} e_1^{k_{i,2}},$$

$$(e_1^{k_{i,1}} e_3^{k_{i,2}})|_{2(k_{i,1}+k_{i,2})}\rho_2 = e_3^{k_{i,1}} e_2^{k_{i,2}}$$

and

$$(e_1^{k_{i,1}}e_3^{k_{i,2}})|_{2(k_{i,1}+k_{i,2})}\rho_3 = e_1^{k_{i,1}}e_3^{k_{i,2}}.$$

Hence the right hand side of (6.3) is equal to

$$\Big\{\{e_2^{k_{1,1}}e_1^{k_{1,2}},\ldots,e_2^{k_{t,1}}e_1^{k_{t,2}}\},\{e_3^{k_{1,1}}e_2^{k_{1,2}},\ldots,e_3^{k_{t,1}}e_2^{k_{t,2}}\},\{e_1^{k_{1,1}}e_3^{k_{1,2}},\ldots,e_1^{k_{t,1}}e_3^{k_{t,2}}\}\Big\}.$$

Thus $|G_{n'}(g_n)| \le 3$ when $g_n \neq 0$. $\square$

Lemma 6.3. Let quasi-elliptic $f^\Psi(z|\tau) = \sum_{\alpha\in M}\psi(\alpha)\theta^\alpha(z|\tau) \in R_2$ and $\beta = (\beta_1,\beta_2,\beta_3,\beta_4) \in \min(M,\psi)$. Suppose

$$\frac{f^\Psi(z|\tau)}{\psi(\beta)\theta^\beta(z)} = \sum_{n=0}^{\infty} d_n(\tau)z^n \text{ with } d_n(\tau)\in\mathbb{K}(\Theta).$$

Let $M = \{\gamma^{(1)},\ldots,\gamma^{(m)}\}$ with $\gamma^{(j)} = (\gamma_1^{(j)},\gamma_2^{(j)},\gamma_3^{(j)},\gamma_4^{(j)})$. For $1\le j\le m$ let

$$\begin{aligned} a_j :=& \frac{\gamma_2^{(j)}-\gamma_4^{(j)}-\beta_2+\beta_4}{2},\\ b_j :=& \frac{\gamma_3^{(j)}-\gamma_4^{(j)}-\beta_3+\beta_4}{2},\\ c_j :=& \gamma_4^{(j)}-\beta_4,\\ r_j :=& \gamma_1^{(j)}-\beta_1, \end{aligned}$$

and

$$t_j := \frac{\psi(\gamma^{(j)})\theta_2(0)^{2a_j}\theta_3(0)^{2b_j}}{\psi(\beta)\theta_1'(0)^{2a_j+2b_j+2c_j}}.$$

For all $n\ge 0$, if $d_n\neq 0$ then

$$|G_n(d_n)| \le |\{\{t_1|_{r_1}\rho,\ldots,t_m|_{r_m}\rho, e_1|_2\rho\} : \rho\in \mathrm{SL}_2(\mathbb{Z})\}|,$$

Proof. First of all we write

$$\frac{f^\Psi(z|\tau)}{\psi(\beta)\theta^\beta(z)} = h_1+\cdots+h_m$$

with $h_j := \frac{\psi(\gamma^{(j)})\theta^{\gamma^{(j)}}(z)}{\psi(\beta)\theta^\beta(z)}$. From the proof of Lemma 5.8 we see that for all $j\in\{1,\ldots,m\}$,

$$W([z^n]h_j(z)) = n$$

and

$$W\left(\frac{\psi(\gamma^{(j)})\theta_2(0)^{2a_j}\theta_3(0)^{2b_j}}{\psi(\beta)\theta_1'(0)^{2a_j+2b_j+2c_j}}\right) = -2a_j-2b_j-3c_j.$$

Then by Proposition 6.1 and expression (6.1) we deduce

$$\left|G_n([z^n]h_j(z))\right| \le \left|\left\{\left\{t_j|_{r_j}\rho,\, e_1|_2\rho\right\} : \rho\in\mathrm{SL}_2(\mathbb{Z})\right\}\right|,$$

where $r_j := -2a_j-2b_j-3c_j = \gamma_1^{(j)}-\beta_1$ following from the definition of a_j,b_j,c_j and $t_j := \frac{\psi(\gamma^{(j)})\theta_2(0)^{2a_j}\theta_3(0)^{2b_j}}{\psi(\beta)\theta_1'(0)^{2a_j+2b_j+2c_j}}$. Consequently, when $d_n\neq 0$ we have $W(d_n)=n$ by Lemma 5.8

and

$$\begin{aligned}|G_n(d_n)| &= \left|\left\{[z^n](h_1(z)+\cdots+h_m(z))\Big|_n \rho\colon \rho\in \mathrm{SL}_2(\mathbb{Z})\right\}\right| \\ &\leq \left|\left\{\left\{[z^n]h_1(z)\Big|_n \rho,\ldots,[z^n]h_m(z)\Big|_n \rho\right\}: \rho\in \mathrm{SL}_2(\mathbb{Z})\right\}\right| \\ &\leq |\{\{t_1|_{r_1}\rho,\ldots,t_m|_{r_m}\rho, e_1|_2\rho\}\colon \rho\in \mathrm{SL}_2(\mathbb{Z})\}|.\end{aligned}$$

□

By the valence formula (Freitag and Busam, 2005, Th. VI.2.3), one can deduce the following result.

Lemma 6.4. Let $q := e^{\pi i\tau}$ and g be a modular form of weight k with a q-expansion $g(q) = \sum_{j=0}^{\infty} v_j q^j$.

$$\text{If } v_j = 0 \text{ for } j \leq \lfloor \frac{k}{6} \rfloor, \text{ then } g = 0.$$

Theorem 6.5. Let $q := e^{\pi i\tau}$, $t_1,\ldots,t_m$, $r_1,\ldots,r_m$ and d_n be the same as in Lemma 6.3, and let

$$\ell := |\{\{t_1|_{r_1}\rho,\ldots,t_m|_{r_m}\rho, e_1|_2\rho\} : \rho\in \mathrm{SL}_2(\mathbb{Z})\}|.$$

For $n \geq 0$ suppose d_n has a q-expansion

$$\sum_{j=0}^{\infty} v_{n,j} q^j.$$

Then

$$d_n = 0 \text{ if and only if } v_{n,j} = 0 \text{ for } j \leq \lfloor \frac{n\ell}{6} \rfloor.$$

Proof. If $d_n(\tau) \equiv \sum_{j=0}^{\infty} v_{n,j} q^j \equiv 0$, it immediately implies that all v_j are zero.

Assume $v_{n,j} = 0$ for $j \leq \lfloor \frac{n\ell}{6} \rfloor$. If $d_n \neq 0$, by Lemma 5.8 we have $W(d_n) = n$ and by Lemma 6.3, $|G_n(d_n)| \leq \ell$. Suppose $G_n(d_n) = \{s_1,\ldots,s_{\ell_n}\}$ and $\ell_n \leq \ell$. Then for every $i \in \{1,\ldots,\ell_n\}$, there exists a unique $j \in \{1,\ldots,\ell_n\}$ such that $s_i|_n S = s_j$; and there exists a unique $k \in \{1,\ldots,\ell_n\}$ such that $s_i|_n T = s_k$. Then

$$\left(\prod_{i=1}^{\ell_n} s_i\right)\Big|_{n\ell_n} S = \prod_{j=1}^{\ell_n} s_j \quad\text{and}\quad \left(\prod_{i=1}^{\ell_n} s_i\right)\Big|_{n\ell_n} T = \prod_{j=1}^{\ell_n} s_j.$$

This yields

$$\left(\prod_{i=1}^{\ell_n} s_i\right)\Big|_{n\ell_n} \rho = \prod_{j=1}^{\ell_n} s_j \quad \text{for all } \rho\in \mathrm{SL}_2(\mathbb{Z}).$$

Moreover, we have proven in Lemma 4.10 of Ye (2017) that $\prod_{j=1}^{\ell_n} s_j$ is a Taylor series in q.

Thus $\prod_{j=1}^{\ell_n} s_j$ is a modular form of weight $n\ell_n$.

Since $\ell_n \leq \ell$ we have $v_{n,j} = 0$ for $j \leq \lfloor \frac{n\ell_n}{6} \rfloor$. By Lemma 6.4, $\prod_{j=1}^{\ell_n} s_j = 0$. Because of the fact that for any meromorphic functions h and g on $\mathbb{H}$, if $(h|_n\rho)(\tau) = g(\tau)$ then $h(\tau) \equiv 0$ if and only if $g(\tau) \equiv 0$, we deduce that s_j must be zero for all $j \in \{1, \ldots, \ell_n\}$, otherwise $s_j \neq 0$ for all $j \in \{1, \ldots, \ell_n\}$ which contradicts $\prod_{j=1}^{\ell_n} s_i = 0$. As $d_n \in G_n(d_n) = \{s_1, \ldots, s_{\ell_n}\}$, we deduce that $d_n = 0$, which contradicts the earlier assumption $d_n \neq 0$. Therefore $d_n = 0$. □

Algorithm 6.6. Let $q = e^{\pi i \tau}$ and $f^{\psi}(z|\tau) = \sum_{\alpha \in M} \psi(\alpha)\theta^{\alpha}(z|\tau) \in R_2$. We have the following algorithm to prove or disprove $f^{\psi}(z|\tau) \equiv 0$.

Input: $f^{\psi} \in R_1$.
Output: True if $f^{\psi} = 0$; False if $f^{\psi} \neq 0$.

Compute the quasi-elliptic components of f^{ψ}, denoted by $f_1, \ldots, f_n$.
Suppose $f_j = \sum_{\alpha \in M_j} \psi(\alpha)\theta^{\alpha}$.
for $j = 1, \ldots, n$ **do**
  take β random in $\min(M_j, \psi)$ and compute the expansion $\frac{f_j(z)}{\theta^{\beta}(z)} = \sum_{k=0}^{\infty} d_k(\tau) z^k$
  suppose $M_j = \{\gamma^{(1)}, \ldots, \gamma^{(m)}\}$
  for $i = 1, \ldots, m$ **do**

  $a_i = \frac{\gamma_2^{(i)} - \gamma_4^{(i)} - \beta_2 + \beta_4}{2}$
  $b_i = \frac{\gamma_3^{(i)} - \gamma_4^{(i)} - \beta_3 + \beta_4}{2}$
  $c_i = \gamma_4^{(i)} - \beta_4$
  $r_i = \gamma_1^{(i)} - \beta_1$
  $t_i = \frac{\psi(\gamma^{(i)})\theta_2(0)^{2a_i}\theta_3(0)^{2b_i}}{\psi(\beta)\theta_1'(0)^{2a_i+2b_i+2c_i}}$
  end for
  $\ell = |\{\{t_1|_{r_1}\rho, \ldots, t_m|_{r_m}\rho, e_1|_2\rho\} : \rho \in \mathrm{SL}_2(\mathbb{Z})\}|$
  for $k = 0, \ldots, \beta_2 + \beta_3 + \beta_4$ **do**
    if $d_k(\tau) \equiv O(q^{\frac{k\ell}{6}+1})$ **then**
      $k \leftarrow k + 1$
    else return False
    end if
  end for
end for
return True

Theorem 6.7. Algorithm 6.6 is correct.

Proof. By Lemma 6.5, $d_k(\tau) \equiv 0$ if and only if $d_k(\tau) \equiv O(q^{\frac{k\ell}{6}+1})$. Since the only difference between Algorithm 3.9 and Algorithm 6.6 is the way in which we check $d_k(\tau) \equiv 0$, it follows that Algorithm 6.6 is correct. □

Example 6.8. (DLMF, 2015, 20.7.1) Prove

$$\theta_2(0)^2\theta_2(z)^2 - \theta_3(0)^2\theta_3(z)^2 + \theta_4(0)^2\theta_4(z)^2 \equiv 0.$$

Proof. Let $\beta := (0,0,0,2)$ and

$$g(z) := \frac{\theta_2(0)^2\theta_2(z)^2}{\theta_4(0)^2\theta_4(z)^2} - \frac{\theta_3(0)^2\theta_3(z)^2}{\theta_4(0)^2\theta_4(z)^2} + 1.$$

Since $g(z)$ is an even function we obtain

$$g(z) = \sum_{k=0}^{\infty} d_{2k}(\tau) z^{2k}$$

with

$$d_0(\tau) = \frac{\theta_2(0)^4 - \theta_3(0)^4 + \theta_4(0)^4}{\theta_4(0)^4}$$

and

$$d_2(\tau) = \frac{\theta_2(0)^3\theta_4(0)\theta_2''(0) - \theta_3(0)^3\theta_4(0)\theta_3''(0) - \theta_2(0)^4\theta_4''(0) + \theta_3(0)^4\theta_4''(0)}{\theta_4(0)^5};$$

and $d_{2k}(\tau)$ $(k > 1)$ are irrelevant to this proof. According to Algorithm 6.6 we need to show that $d_0(\tau) = O(q)$ and $d_2(\tau) = O(q^{\frac{\ell}{3}+1})$ where

$$\ell = \left| \left\{ 1, -\frac{\theta_3^4}{\theta_2^4}\Big|_0 \rho, \frac{\theta_4^2\theta_1'^2}{\theta_2^6\theta_3^2}\Big|_0 \rho, e_1|_2\rho \colon \rho \in \mathrm{SL}_2(\mathbb{Z}) \right\} \right|.$$

Using Lemmas 5.6 and 5.7, one verifies that $\ell = 6$ and the corresponding elements are

$$\left\{1, -\frac{\theta_3^4}{\theta_2^4}, \frac{\theta_4^2\theta_1'^2}{\theta_2^6\theta_3^2}, e_1\right\}, \left\{1, -\frac{\theta_3^4}{\theta_4^4}, \frac{\theta_2^2\theta_1'^2}{\theta_4^6\theta_3^2}, e_2\right\}, \left\{1, \frac{\theta_4^4}{\theta_2^4}, -\frac{\theta_3^2\theta_1'^2}{\theta_2^6\theta_4^2}, e_1\right\},$$

$$\left\{1, -\frac{\theta_4^4}{\theta_3^4}, -\frac{\theta_2^2\theta_1'^2}{\theta_3^6\theta_4^2}, e_3\right\}, \left\{1, \frac{\theta_2^4}{\theta_4^4}, -\frac{\theta_3^2\theta_1'^2}{\theta_4^6\theta_2^2}, e_2\right\}, \left\{1, -\frac{\theta_2^4}{\theta_3^4}, -\frac{\theta_4^2\theta_1'^2}{\theta_3^6\theta_2^2}, e_3\right\}.$$

By Definition 1.1 we have $d_0(\tau) = O(q)$ and $d_2(\tau) = O(q^3)$. □

Speed comparison. The only difference between Algorithms 3.9 and 6.6 is the way of dealing with the coefficients in the series expansion $\sum_{k=0}^{\infty} d_k(\tau) z^k$; namely, to check if certain $d_k(\tau)$ are zero. In Algorithm 6.6 we do this by computing the orbit

$$\{t_1|_{r_1}\rho, \ldots, t_m|_{r_m}\rho, e_1|_2\rho : \rho \in \mathrm{SL}_2(\mathbb{Z})\},$$

which is needed for the orbit length ℓ. The t_j do not contain any of $\theta_j^{(k)}$ $(k \geq 1)$, except for θ_1'. All of $\theta_2, \theta_3, \theta_4$ and θ_1' have very simple modular transformations.[4] In contrast, Algorithm 3.9 uses Algorithm 5.11 of Ye (2017) and it directly computes the leading term

[4] See Lemmas 5.6 and 5.7.

orbits of certain $d_k(\tau)$, which contains $\theta_j^{(k)}$ ($k \geq 1$) with sophisticated modular transformations.[5] In addition, the coefficients $d_k(\tau)$ become more and more complicated when the degree of z grows. Thus Algorithm 3.9 needs more time on the orbit computation than Algorithm 6.6, especially when the identity we want to prove contains a large input. More details can be found in Section 6.3 of Ye (2016).

7. Conclusion

In the literature, not many high degree identities in R_1 and R_2 are found. The one with the highest degree we were able to find in R_1 is identity (1.3) in Chapter 2, whilst we have a way of producing all relations in R_1, which can be found in Chapter 6 of Ye (2016). Moreover, we are preparing a paper that determines the generators of the ideal containing all relations in R_2.

On the other hand, based on this article, algorithmically dealing with other types of identities becomes possible. For instance, we have algorithms to prove identities like

$$\theta_2\,\theta_3\,\theta_4\,\theta_1(2z,q) - 2\theta_1(z)\theta_2(z)\theta_3(z)\theta_4(z) \equiv 0,$$

an identity from (Whittaker and Watson, 1927, p. 485) and

$$\sum_{j=1}^{4} \theta_j(x)\theta_j(y)\theta_j(u)\theta_j(v) - 2\theta_3(x_1)\theta_3(y_1)\theta_3(u_1)\theta_3(v_1) \equiv 0,$$

an identity from (Mumford, 1983, p. 17), where $x_1 := \frac{1}{2}(x+y+u+v)$ and $y_1 := \frac{1}{2}(x+y-u-v)$, $u_1 := \frac{1}{2}(x-y+u-v)$ and $v_1 := \frac{1}{2}(x-y-u+v)$. As one of the anonymous referees pointed out, an interesting discussion of the identity has been given recently by Koornwinder (2014).

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[5] See Corollary 3.5 of Ye (2017).

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