Reduction-Based Creative Telescoping for Algebraic Functions*

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ABSTRACT

Continuing a series of articles in the past few years on creative telescoping using reductions, we develop a new algorithm to construct minimal telescopers for algebraic functions. This algorithm is based on Trager's Hermite reduction and on polynomial reduction, which was originally designed for hyperexponential functions and extended to the algebraic case in this paper.

Keywords

Algebraic function, Integral basis, Trager's Reduction, Telescoper

1. INTRODUCTION

The classical question in symbolic integration is whether the integral of a given function can be written in "closed form". In its most restricted form, the question is whether for a given function f belonging to some domain D there exists another function g, also belonging to D, such that $f = g'$. For example, if D is the field of rational functions, then for $f = 1/x^2$ we can find $g = -1/x$, while for $f = 1/x$ no suitable g exists. When no g exists in D, there are several other questions we may ask. One possibility is to ask whether there is some extension E of D such that in E there exists some g with $g' = f$. For example, in the case of elementary functions, Liouville's principle restricts the possible extensions E, and there are algorithms which construct such extensions whenever possible. Another possibility is to ask whether for some modification $\tilde{f} \in D$ of f there exists a $g \in D$ such that $\tilde{f} = g'$. Creative telescoping is a question of this type. Here we are dealing with domains D containing functions in several variables, say x and t, and the question is whether there is a linear differential operator P, nonzero and free of x, such that there exists a $g \in D$ with $P \cdot f = g'$, where g' denotes the derivative of g with respect to x. Typically, g itself has the form $Q \cdot f$ for some operator Q (which may be zero and need not be free of x). In this case, we call P a telescoper for f, and Q a certificate for P.

Creative telescoping is the backbone of definite integration, because $P \cdot f = (Q \cdot f)'$ implies, for instance, $P \cdot \int_0^1 f(x,t)dx = (Q \cdot f)(1) - (Q \cdot f)(0)$. A telescoper P for f thus gives rise to an annihilating operator for the definite integral $F(t) = \int_0^1 f(x,t)dx$.

Example 1 (Manin [20])**.** *The algebraic function*

$$f(x,t) = \frac{1}{\sqrt{x(x-1)(x-t)}}$$

does not admit an elementary integral with respect to x. However, we have $P \cdot f = (Q \cdot f)'$ for

$$P = 4(t-1)t\,D_t^2 + 4(2t-1)D_t + 1, \quad Q = \frac{2x(x-1)}{t-x}.$$

This implies

$$P \cdot \int_0^1 f(x,t)dx = \Big[\frac{2x(x-1)}{t-x} f(x,t)\Big]_{x=0}^{x=1},$$

so the integral $F(t) = \int_0^1 \frac{1}{\sqrt{x(x-1)(x-t)}}dx$ satisfies the differential equation

$$4(t-1)t\,F''(t) + 4(2t-1)F'(t) + F(t) = 0.$$

In the common case when the right hand side collapses to zero, we say that the integral has "natural boundaries". Readers not familiar with creative telescoping are referred to the literature [21, 25, 27, 26, 18, 16] for additional motivation, theory, algorithms, implementations, and applications.

There are several ways to find telescopers for a given $f \in D$. In recent years, an approach has become popular

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which has the feature that it can find a telescoper without also constructing the corresponding certificate. This is interesting because certificates tend to be much larger than telescopers, and in some applications, for instance when an integral has natural boundaries, only the telescoper is of interest. This approach was first formulated for rational functions $f \in C(t,x)$ in [1] and later generalized to rational functions in several variables [3, 19], to hyperexponential functions [2] and, for the shift case, to hypergeometric terms [7, 14] and binomial sums [4]. In the present paper, we will extend the approach to algebraic functions.

The basic principle of the general approach is as follows. Assume that the x-constants $\mathrm{Const}_x(D) = \{ c \in D : c' = 0 \}$ form a field and that D is a vector space over the field of x-constants. Assume further that there is some $\mathrm{Const}_x(D)$-linear map $[\cdot]\colon D \to D$ such that for every $f \in D$ there exists a $g \in D$ with $f - [f] = g'$. Such a map is called a *reduction*. For example, in $D = C(t,x)$ Hermite reduction [13] produces for every $f \in D$ some $g \in D$ such that $f - g'$ is either zero or a rational function with a square-free denominator. In this case, we can take $[f] = f - g'$. In order to find a telescoper, we can compute $[f]$, $[\partial_t \cdot f]$, $[\partial_t^2 \cdot f]$, ..., until we find that they are linearly dependent over $\mathrm{Const}_x(D)$. Once we find a relation $p_0[f] + \cdots + p_r[\partial_t^r \cdot f] = 0$, then, by linearity, $[p_0 f + \cdots + p_r \partial_t^r \cdot f] = 0$, and then, by definition of $[\cdot]$, there exists a $g \in D$ such that $(p_0 + \cdots + p_r \partial_t^r) \cdot f = g'$. In other words, $P = p_0 + \cdots + p_r \partial_t^r$ is a telescoper.

There are two ways to guarantee that this method terminates. The first requires that we already know for other reasons that a telescoper exists. The idea is then to show that the reduction $[\cdot]$ has the property that when $f \in D$ is such that there exists a $g \in D$ with $g' = f$, then $[f] = 0$. If this is the case and $P = p_0 + \cdots + p_r \partial_t^r$ is a telescoper for f, then $P \cdot f$ is integrable in D, so $[P \cdot f] = 0$, and by linearity $[f]$, ..., $[\partial_t^r \cdot f]$ are linearly dependent over $\mathrm{Const}_x(D)$. This means that the method won't miss any telescoper. In particular, this argument has the nice feature that we are guaranteed to find a telescoper of smallest possible order r. This approach was taken in [7]. The second way consists in showing that the $\mathrm{Const}_x(D)$-vector space generated by $\{ [\partial_t^i \cdot f] : i \in \mathbb{N} \}$ has finite dimension. This approach was taken in [1, 2]. It has the nice additional feature that every bound for the dimension of this vector space gives rise to a bound for the order of the telescoper. In particular, it implies the existence of a telescoper.

In this paper, we show that Trager's Hermite reduction for algebraic functions directly gives rise to a reduction-based creative telescoping algorithm via the first approach (Section 4). We will combine Trager's Hermite reduction with a second reduction, called polynomial reduction (Section 5), to obtain a reduction-based creative telescoping algorithm for algebraic functions via the second approach (Section 6). This gives a new bound for the order of the telescopers, and in particular an independent proof for their existence.

A few years ago, Chen et al. [9] have already considered the problem of creative telescoping for algebraic functions. They have pointed out that by canceling residues of the integrand, a given creative telescoping problem can be reduced to a creative telescoping problem for a function with no residues, which may be much smaller than the original function. For this smaller function, however, they still need to construct a certificate. With some regularity assumption on certificates, Bostan et al. [3] gave a telescoping algorithm for multivariate rational functions using Griffiths-Dwork reduction, which then leads to a reduction-based telescoping algorithm for algebraic functions by Theorem 6 of [9]. The algorithms presented in the present paper can find minimal telescopers for bivariate algebraic functions without any regularity assumption on certificates. Our results also translate into a certificate-free creative telescoping algorithm for rational functions in three variables.

2. ALGEBRAIC FUNCTIONS

Throughout the paper, let C be a field of characteristic zero, $K = C(t)$, and $\bar{K}$ the algebraic closure of K. We consider algebraic functions over K. For some absolutely irreducible polynomial $m \in K[x,y]$, we consider the field $A = K(x)[y]/\langle m \rangle$. If $n = \deg_y m$, then every element of A can be written uniquely in the form $f = f_0 + f_1 y + \cdots + f_{n-1} y^{n-1}$ for some $f_0, \ldots, f_{n-1} \in K(x)$.

The element $y \in A$ is a solution of the equation $m = 0$, because in A we have $m = 0$ by construction. The polynomial m also admits n distinct solutions in the field

$$\bar{K}\langle\langle x-a \rangle\rangle := \bigcup_{r \in \mathbb{N}\setminus\{0\}} \bar{K}((\ (x-a)^{1/r} \ ))$$

of formal Puiseux series around $a \in \bar{K}$. There are also n distinct solutions in the field

$$\bar{K}\langle\langle x^{-1} \rangle\rangle := \bigcup_{r \in \mathbb{N}\setminus\{0\}} \bar{K}((x^{-1/r}))$$

of formal Puiseux series around ∞. Since $\bar{K}\langle\langle x^{-1}\rangle\rangle$ and the $\bar{K}\langle\langle x-a\rangle\rangle$ are fields, we can associate to every $f \in A$ and every $a \in \bar{K} \cup \{\infty\}$ in a natural way n distinct series objects with fractional exponents, by plugging any of the n distinct series solutions of m into the representation $f = f_0 + \cdots + f_{n-1}y^{n-1}$. In other words, for every $a \in \bar{K} \cup \{\infty\}$ there are n distinct natural ring homomorphisms from A to $K\langle\langle x-a\rangle\rangle$ or $K\langle\langle x^{-1}\rangle\rangle$, respectively.

In the field A as well as the fields $\bar{K}\langle\langle x-a\rangle\rangle$ and $\bar{K}\langle\langle x^{-1}\rangle\rangle$, we have natural differentiations with respect to x. For a series, differentiation is defined termwise using the usual rules $\big((x-a)^{\nu+n}\big)' = (\nu+n)(x-a)^{\nu+n-1}$ and $\big((x^{-1})^{\nu+n}\big)' = -(\nu+n)(x^{-1})^{\nu+n+1}$. For the elements of A, note first that $m(x,y) = 0$ implies

$$m(x,y)' = (\frac{d}{dx}m)(x,y) + (\frac{d}{dy}m)(x,y)y' = 0, \qquad (1)$$

so $y' = -(\frac{d}{dx}m)(x,y)/(\frac{d}{dy}m)(x,y)$. Regarding m as element of $K(x)[y]$ and observing that $0 < \deg_y \frac{d}{dy}m < n$, we have $\gcd(m, \frac{d}{dy}m) = 1$ in $K(x)[y]$, so $\frac{d}{dy}m$ is invertible in $A = K(x)[y]/\langle m \rangle$. Note that we have $x' = 1$ and $c' = 0$ for all $c \in K = C(t)$, in particular also $t' = 0$. The derivative of an arbitrary element $f \in A$, say $f = p(x,y)$ for some $p \in K(x)[y]$ of degree less than n, is

$$f' = (\frac{d}{dx}p)(x,y) + (\frac{d}{dy}p)(x,y)y'.$$

Thus we have an action of the algebra $K(x)[\partial_x]$ of differential operators on A.

The derivations on A and on the series domains are compatible in the sense that for every $f \in A$, the series associated to f' are precisely the derivatives of the series associated to f.

In the context of creative telescoping, we will also need to differentiate with respect to t. The action of $K(x)[\partial_x]$ on A and on the series domains is extended to an action of $K(x)[\partial_x, \partial_t]$ on A and on the series domains. On A, the action of ∂_t is defined as the unique derivation with $\partial_t \cdot t = 1$ and $\partial_t \cdot x = 0$, analogously to the construction above. For the series domains, ∂_t acts on the coefficients (which are elements of $\bar{K}$) in the natural way, and does not affect x. Since each particular element $c \in \bar{K}$ belongs to a finite algebraic extension of K, the result $\partial_t \cdot c$ is uniquely determined. The actions of the larger operator algebra $K(x)[\partial_x, \partial_t]$ on A and on the series domains are compatible to each other.

In this paper, the notation f' will always refer to the derivative $\partial_x \cdot f$ with respect to x, not with respect to t.

Trager's Hermite reduction for algebraic functions rests on the notion of integral bases. Let us recall the relevant definitions and properties. Although the elements of a Puiseux series ring $\bar{K}\langle\langle x-a\rangle\rangle$ are formal objects, the series notation suggests certain analogies with complex functions. Terms $(x-a)^\alpha$ or $(\frac{1}{x})^\alpha$ are called *integral* if $\alpha \geq 0$. A series in $\bar{K}\langle\langle x-a\rangle\rangle$ or $\bar{K}\langle\langle x^{-1}\rangle\rangle$ is called integral if it only contains integral terms. A non-integral series is said to have a *pole* at the reference point. Note that in this terminology also $1/\sqrt{x}$ has a pole at 0. Note also that the terminology only refers to x but not to t.

Integrality at $a \in \bar{K}$ is not preserved by differentiation, but if f is integral at a, then so is $(x-a)f'$. On the other hand, integrality at infinity is preserved by differentiation, we even have the stronger property that when f is integral at infinity, then not only f' but also $xf' = (x^{-1})^{-1}f'$ is integral at infinity.

An element $f \in A = K(x)[y]/\langle m\rangle$ is called (locally) integral at $a \in \bar{K} \cup \{\infty\}$ if for every series associated to y the corresponding series for f is integral. The element f is called (globally) integral if it is locally integral at every $a \in \bar{K}$ ("at all finite places"). This is the case if and only if the minimal polynomial of f in $K[x, y]$ is monic with respect to y. Because of Chevalley's theorem [10, page 9, Corollary 3], any non-constant algebraic function has at least one pole. Equivalently, an element f is integral at all $a \in \bar{K} \cup \{\infty\}$ if and only if it is constant.

For an element $f \in A$ to have a "pole" at $a \in \bar{K} \cup \{\infty\}$ means that f is not locally integral at a; to have a "double pole" at a means that $(x-a)f$ (or $\frac{1}{x}f$ if $a = \infty$) is not integral; to have a "double root" at a means that $f/(x-a)^2$ (or $f/(\frac{1}{x})^2 = x^2 f$ if $a = \infty$) is integral, and so on.

The set of all globally integral elements $f \in A$ forms a $K[x]$-submodule of A. A basis $\{\omega_1, \ldots, \omega_n\}$ of this module is called an *integral basis* for A. Such bases exist, and algorithms are known for computing them [24, 22, 15]. For a fixed $a \in \bar{K}$, let $\bar{K}(x)_a$ be the ring of rational functions p/q with $q(a) \neq 0$, and write $\bar{K}(x)_\infty$ for the ring of all rational functions p/q with $\deg_x(p) \leq \deg_x(q)$. Then the set of all $f \in A$ which are locally integral at some fixed $a \in \bar{K} \cup \{\infty\}$ forms a $\bar{K}(x)_a$-module. A basis of this module is called a *local integral basis* at a for A. Also local integral bases can be computed.

An integral basis $\{\omega_1, \ldots, \omega_n\}$ is always also a $K(x)$-vector space basis of A. A key feature of integral bases is that they make poles explicit. Writing an element $f \in A$ as a linear combination $f = \sum_{i=1}^n f_i\omega_i$ for some $f_i \in K(x)$, we have that f has a pole at $a \in \bar{K}$ if and only if at least one of the f_i has a pole there.

Lemma 2. *Let $\{\omega_1, \ldots, \omega_n\}$ be a local integral basis of A at $a \in \bar{K} \cup \{\infty\}$. Let $f \in A$ and $f_1, \ldots, f_n \in K(x)$ be such that $f = \sum_{i=1}^n f_i\omega_i$. Then f is integral at a if and only if each $f_i\omega_i$ is integral at a.*

Proof. The direction "⇐" is obvious. To show "⇒", suppose that f is integral at a. Then there exist $w_1, \ldots, w_n \in \bar{K}(x)_a$ such that $f = \sum_{i=1}^n w_i\omega_i$. Thus $\sum_{i=1}^n (w_i - f_i)\omega_i = 0$, and then $w_i = f_i$ for all i, because $\omega_1, \ldots, \omega_n$ is a vector space basis of A. As elements of $\bar{K}(x)_a$, the f_i are integral at a, and hence also all the $f_i\omega_i$ are integral at a. ■

The lemma says in particular that poles of the f_i in a linear combination $\sum_{i=1}^n f_i\omega_i$ cannot cancel each other.

Lemma 3. *Let $\{\omega_1, \ldots, \omega_n\}$ be an integral basis of A. Let $e \in K[x]$ and $M = ((m_{i,j}))_{i,j=1}^n \in K[x]^{n\times n}$ be such that*

$$e\,\omega_i' = \sum_{j=1}^n m_{i,j}\omega_j$$

for $i = 1, \ldots, n$ and $\gcd(e, m_{1,1}, \ldots, m_{n,n}) = 1$. Then e is squarefree.

Proof. Let $a \in \bar{K}$ be a root of e. We show that a is not a multiple root. Since ω_i is integral, it is in particular locally integral at a. Therefore $(x-a)\omega_i'$ is locally integral at a. Since $\omega_1, \ldots, \omega_n$ is an integral basis, it follows that $(x-a)m_{i,j}/e \in \bar{K}(x)_a$ for all i, j. Because of $\gcd(e, m_{1,1}, \ldots, m_{n,n}) = 1$, no factor $x-a$ of e can be canceled by all the $m_{i,j}$. Therefore the factor $x-a$ can appear in e only once. ■

Lemma 4. *Let $\{\omega_1, \ldots, \omega_n\}$ be a local integral basis at infinity of A. Let $e \in K[x]$ and $M = ((m_{i,j}))_{i,j=1}^n \in K[x]^{n\times n}$ be defined as in Lemma 3. Then $\deg_x(m_{i,j}) < \deg_x(e)$ for all i, j.*

Proof. Since every ω_i is locally integral at infinity, so is every $x\,\omega_i'$. Since $\omega_1, \ldots, \omega_n$ is an integral basis at infinity, it follows that $xm_{i,j}/e \in \bar{K}(x)_\infty$ for all i, j. This means that $1 + \deg_x(m_{i,j}) \leq \deg_x(e)$ for all i, j, and therefore $\deg_x(m_{i,j}) < \deg_x(e)$, as claimed. ■

A $K(x)$-vector space basis $\{\omega_1, \ldots, \omega_n\}$ of A is called *normal* at $a \in \bar{K} \cup \{\infty\}$ if there exist $r_1, \ldots, r_n \in K(x)$ such that $\{r_1\omega_1, \ldots, r_n\omega_n\}$ is a local integral basis at a. Trager shows how to construct an integral basis which is normal at infinity from a given integral basis and a given local integral basis at infinity [24].

Although normality is a somewhat weaker condition on a basis than integrality, it also excludes the possibility that poles in the terms of a linear combination of basis elements can cancel:

Lemma 5. *Let $\{\omega_1, \ldots, \omega_n\}$ be a basis of A which is normal at some $a \in \bar{K} \cup \{\infty\}$. Let $f = \sum_{i=1}^n f_i\omega_i$ for some $f_1, \ldots, f_n \in K(x)$. Then f has a pole at a if and only if there is some i such that $f_i\omega_i$ has a pole at a.*

Proof. Let $r_1, \ldots, r_n \in K(x)$ be such that $r_1\omega_1, \ldots, r_n\omega_n$ is a local integral basis at a. By $f = \sum_{i=1}^n (f_i r_i^{-1})(r_i\omega_i)$ and by Lemma 2, f is integral at a iff all $f_i r_i^{-1} r_i\omega_i = f_i\omega_i$ are integral at a. ■

3. HERMITE REDUCTION

We now recall the Hermite reduction for algebraic functions [24, 12, 6]. Let $\{\omega_1,\dots,\omega_n\}$ be an integral basis for A. Further let $e, m_{i,j} \in K[x]$ $(1 \le i,j \le n)$ be such that $e\omega_i' = \sum_{j=1}^n m_{i,j}\omega_i$ and $\gcd(e, m_{1,1}, m_{1,2}, \dots, m_{n,n}) = 1$. For describing the Hermite reduction we fix an integrand $f \in A$ and represent it in the integral basis, i.e., $f = \sum_{i=1}^n (f_i/D)\,\omega_i$ with $D, f_1, \dots, f_n \in K[x]$. The purpose is to find $g, h \in A$ such that $f = g' + h$ and $h = \sum_{i=1}^n (h_i/D^*)\,\omega_i$ with $h_1, \dots, h_n \in K[x]$ and D^* denoting the squarefree part of D. As differentiating the ω_i can introduce denominators, namely the factors of e, it is convenient to consider those denominators from the very beginning on, which means that we shall assume $e \mid D$. Note that $\gcd(D, f_1, \dots, f_n)$ can then be nontrivial. Let $v \in K[x]$ be a nontrivial squarefree factor of D of multiplicity $\mu > 1$. Then $D = uv^\mu$ for some $u \in K[x]$ with $\gcd(u, v) = 1$ and $\gcd(v, v') = 1$. One step of the Hermite reduction is as follows:

$$\sum_{i=1}^n \frac{f_i}{uv^\mu}\omega_i = \left(\sum_{i=1}^n \frac{g_i}{v^{\mu-1}}\omega_i\right)' + \sum_{i=1}^n \frac{h_i}{uv^{\mu-1}}\omega_i, \qquad (2)$$

where $g_i, h_i \in K[x]$ and $\deg_x(g_i) < \deg_x(v)$. The existence of such g_i's and h_i's follows from the crucial fact that the elements $s_i := uv^\mu(v^{1-\mu}\omega_i)'$ with $i \in \{1,\dots,n\}$ form a local integral basis at each root of v [24, page 46]. By a repeated application of such reduction steps, one can decompose any $f \in A$ as $f = g' + h$ where the denominators of the coefficients of h are squarefree and the coefficients of g are proper rational functions (i.e., their numerators have smaller degree than their denominators).

It was observed that Hermite reduction itself often takes less time than the construction of an integral basis. If Hermite reduction is applied to some other basis, for instance the standard basis $\{1, y, \dots, y^{n-1}\}$, it either succeeds or it runs into a division by zero. Bronstein [5] noticed that when a division by zero occurs, then the basis can be replaced by some other basis that is a little closer to an integral basis, just as much as is needed to avoid this particular division by zero. After finitely many such basis changes, the Hermite reduction will come to an end and produce a correct output. This variant is known as lazy Hermite reduction.

4. TELESCOPING VIA REDUCTIONS: FIRST APPROACH

Recall from the introduction that reduction-based creative telescoping requires some K-linear map $[\cdot]\colon A \to A$ with the property that $f - [f]$ is integrable in A for every $f \in A$. This is sufficient for the correctness of the method, but additional properties are needed in order to ensure that the method terminates.

As also explained already in the introduction, one possibility consists in showing that $[f] = 0$ whenever f is integrable. Trager showed that his Hermite reduction has this property [24, page 50, Theorem 1]. As this result was never published elsewhere, and for the sake of completeness, we reproduce his proof here.

Lemma 6. *Let $W = \{\omega_1,\dots,\omega_n\}$ be an integral basis for A that is normal at infinity. Let $g = \sum_{i=1}^n g_i\omega_i \in A$ be such that all its coefficients $g_i \in K(x)$ are proper rational functions. If an integral element $f \in A$ has a pole at infinity, then also $f + g$ has a pole at infinity.*

Proof. Since f is assumed to be integral we can write it as $f = f_1\omega_1 + \dots + f_n\omega_n$ with $f_i \in K[x]$. If f has a pole at infinity, there is at least one index i such that $f_i\omega_i$ has a pole at infinity. There are two cases why this can happen.

(a) The polynomial f_i has positive degree. This means that $f_i + g_i$ has a pole at infinity, because the g_i are proper rational functions. Thus $(f_i + g_i)\omega_i$ has a pole at infinity, because ω_i has no poles at finite places and therefore no root at infinity.

(b) The integral basis element ω_i is not constant and f_i is not zero. Hence ω_i has a pole at infinity, and this also implies that $(f_i + g_i)\omega_i$ has a pole at infinity, again employing the fact that g_i is a proper rational function.

In both cases, therefore, $f + g = \sum_{i=1}^n (f_i + g_i)\omega_i$ has a pole at infinity by Lemma 5. ∎

Theorem 7. *Suppose that $f \in A$ has at least a double root at infinity (i.e., every series in $\bar{K}\langle\langle x^{-1}\rangle\rangle$ associated to f only contains monomials $(1/x)^\alpha$ with $\alpha \ge 2$). Let $W = \{\omega_1,\dots,\omega_n\}$ be an integral basis for A that is normal at infinity. If $f = g' + h$ is the result of the Hermite reduction with respect to W, then $h = 0$ if and only if f is integrable in A.*

Proof. The direction "⇒" is trivial. To show the implication "⇐" assume that f is integrable in A. From $f = g' + h$ it follows that then also h is integrable in A; let $H \in A$ be such that $H' = h$. In order to show that $h = 0$, we show that H is constant. To this end, it suffices to show that it has neither finite poles nor a pole at infinity; the claim then follows from Chevalley's theorem.

It is clear that H has no finite poles because h has at most simple poles (i.e., all series associated to h have only exponents ≥ -1). This follows from the facts that the ω_i are integral and that the coefficients of h have squarefree denominators.

If H has a pole at infinity, then by Lemma 6 also $g + H$ must have a pole at infinity, because Hermite reduction produces $g = \sum_i g_i\omega_i$ with proper rational functions g_i. On the other hand, since $f = g' + h = (g + H)'$ has at least a double root at infinity by assumption, $g + H$ must have at least a single root at infinity. This is a contradiction. ∎

Note that the condition in Theorem 7 that f has a double root at infinity is not a restriction at all, as it can always be achieved by a suitable change of variables. Let $a \in C$ be a regular point; this means that all series in $\bar{K}\langle\langle x - a\rangle\rangle$ associated to f are formal power series. By the substitution $x \to a + 1/x$ the regular point a is moved to infinity. From

$$\int f(x)\,\mathrm{d}x = \int f\left(\frac{1}{x} + a\right)\left(-\frac{1}{x^2}\right)\mathrm{d}x$$

we see that the new integrand has a double root at infinity.

Moreover, since the action of ∂_t on series domains is defined coefficient-wise, it follows that when f has at least a double root at infinity (with respect to x), then this is also true for $\partial_t\cdot f, \partial_t^2\cdot f, \partial_t^3\cdot f, \dots$, and then also for every K-linear combination $p_0 f + p_1\partial_t\cdot f + \dots + p_r\partial_t^r\cdot f$. Thus Theorem 7 implies that $p_0 + p_1\partial_t + \dots + p_r\partial_t^r$ is a telescoper for f if *and only if* $[p_0 f + p_1\partial_t\cdot f + \dots + p_r\partial_t^r\cdot f] = 0$.

We already know for other reasons [26, 11, 9] that telescopers for algebraic functions exist, and therefore the reduction-based creative telescoping procedure with Hermite reduction with respect to an integral basis that is normal at infinity as reduction function succeeds when applied to an integrand $f \in A$ that has a double root at infinity. In particular, the method finds a telescoper of smallest possible order. Again, if f has no double root at infinity, we can produce one by a change of variables. Note that a change of variables $x \to a + 1/x$ with $a \in C$ has no effect on the telescoper.

Example 8. *We consider the algebraic function $f = y/x^2$ where y is a solution of the third-degree polynomial equation $m(x,y) = y^3 + y + x + t = 0$. An integral basis for $A = K(x)[y]/\langle m\rangle$ that is normal at infinity is given by $\omega_1 = 1$, $\omega_2 = y$, $\omega_3 = y^2$. (This means that employing lazy Hermite reduction avoids completely the computation of an integral basis in this example.)*

By solving Equation (1) for y' we obtain

$$y' = \frac{-6y^2 + 9(t+x)y - 4}{27x^2 + 54tx + 27t^2 + 4}.$$

Then for the differentiation matrix $\frac{1}{e}M$, a simple calculation yields

$$\begin{pmatrix}\omega_1'\\ \omega_2'\\ \omega_3'\end{pmatrix} = \frac{1}{e}\begin{pmatrix} 0 & 0 & 0\\ -4 & 9(t+x) & -6\\ 12(t+x) & 4 & 18(t+x)\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}$$

with $e = 27x^2 + 54xt + 27t^2 + 4$. Thus we write $f = \sum_{i=1}^{3}(f_i/D)\omega_i$ with $f_1 = f_3 = 0$, $f_2 = e$, and $D = x^2 e$. After a single step the Hermite reduction delivers the result

$$f = \underbrace{\Big(-\frac{y}{x}\Big)'}_{=g_0} + \underbrace{\frac{-6y^2 + 9(x+t)y - 4}{x(27x^2 + 54xt + 27t^2 + 4)}}_{=h_0}.$$

As the Hermite remainder h_0 is nonzero, Theorem 7 tells us that f is not integrable in A. Hence we continue by applying Hermite reduction to

$$\partial_t \cdot f = \frac{-6y^2 + 9(x+t)y - 4}{x^2(27x^2 + 54xt + 27t^2 + 4)}.$$

Note that we could as well take $\partial_t \cdot h_0$ instead of $\partial_t \cdot f$, which in general should result in a faster algorithm. Again after a single reduction step, the decomposition $\partial_t \cdot f = g_1' + h_1$ is obtained, where

$$g_1 = \frac{6y^2 - 9ty + 4}{x(27t^2 + 4)}$$

$$h_1 = \frac{6\big((9x + 27t)y^2 - (27xt + 27t^2 - 2)y + 6x + 18t\big)}{x(27t^2 + 4)(27x^2 + 54xt + 27t^2 + 4)}.$$

Since h_0 and h_1 are linearly independent over $K = C(t)$, we continue with $\partial_t^2 \cdot f$. This time however, it is preferable to start the Hermite reduction with $\partial_t \cdot h_1$, which is given by

$$\frac{1}{x(27t^2 + 4)^2(27x^2 + 54xt + 27t^2 + 4)^2}.$$

Setting $v = 27x^2 + 54xt + 27t^2 + 4 = e$ and doing one reduction step, the Hermite remainder h_2 is found to be

$$\big(6\big((-729xt - 1539t^2 + 96)y^2 + (1215xt^2 - 144x + 1215t^3 - 306t)y - 486xt - 1026t^2 + 64\big)\big) \;/\; \big(x(27t^2 + 4)^2 e\big).$$

The corresponding integrable part g_2 is not displayed here for space reasons.

Now one can find a linear dependence between h_0, h_1, h_2 that gives rise to the telescoper $(27t^2 + 4)\partial_t^2 + 81t\partial_t + 24$, which is indeed the minimal one for this example.

5. POLYNOMIAL REDUCTION

Recall that instead of requesting that $[f] = 0$ if and only if f is integrable (first approach), we can also justify the termination of reduction-based creative telescoping by showing that the K-vector space generated by $\{\,[\partial_t^i f] : i \in \mathbb{N}\,\}$ has finite dimension (second approach). If $[\cdot]$ is just the Hermite reduction, we do not necessarily have this property. We therefore introduce below an additional reduction, called *polynomial reduction,* which we apply after Hermite reduction. We then show that the combined reduction (Hermite reduction followed by polynomial reduction) has the desired dimension property for the space of remainders. As a result, we obtain a new bound on the order of the telescoper, which is similar to those in [9, 8].

In this approach, we use two integral bases. First we use a global integral basis in order to perform Hermite reduction. Then we write the remainder h with respect to some local integral basis at infinity and perform the polynomial reduction on this representation.

Throughout this section let $W = (\omega_1, \dots, \omega_n)^T \in A^n$ be such that $\{\omega_1, \dots, \omega_n\}$ is a global integral basis of A, and let $e \in K[x]$ and $M = (m_{i,j}) \in K[x]^{n\times n}$ be such that $eW' = MW$ and $\gcd(e, m_{1,1}, m_{1,2}, \dots, m_{n,n}) = 1$. The Hermite reduction described in Section 3 decomposes an input element $f \in A$ into the form

$$f = g' + h = g' + \sum_{i=1}^{n} \frac{h_i}{de}\omega_i, \qquad g, h \in A,$$

with $h_i, d \in K[x]$ such that $\gcd(d, e) = \gcd(h_i, de) = 1$ and d is squarefree.

Lemma 9. *If h is integrable in A, then d is in K.*

Proof. Suppose that h is integrable in A, i.e., there exist $a, b_i \in K[x]$ such that $h = \big(\frac{1}{a}\sum_{i=1}^{n} b_i\omega_i\big)'$. Then

$$h = \sum_{i=1}^{n} \frac{h_i}{de}\omega_i = \sum_{i=1}^{n}\Big(\Big(\frac{b_i}{a}\Big)'\omega_i + \frac{b_i}{ae}\sum_{j=1}^{n} m_{i,j}\omega_j\Big).$$

We show that a is constant. Otherwise, for any irreducible factor p of a, we would have that h has a pole of multiplicity greater than 1 at the roots of p. This contradicts the fact that d, e are squarefree. Thus, d is a constant. ∎

By the extended Euclidean algorithm, we compute $u_i, v_i \in K[x]$ such that $h_i = u_i d + v_i e$ and $\deg_x(v_i) < \deg_x(d)$. Then the Hermite remainder h decomposes as

$$\sum_{i=1}^{n} \frac{h_i}{de}\omega_i = \sum_{i=1}^{n} \frac{u_i}{e}\omega_i + \sum_{i=1}^{n} \frac{v_i}{d}\omega_i. \tag{3}$$

We now introduce the *polynomial reduction* whose goal is to confine the u_i to a finite-dimensional vector space over K. Similar reductions have been introduced and used in creative telescoping for hyperexponential functions [2] and hypergeometric terms [7]. Let $V = (\nu_1, \dots, \nu_n)^T \in A^n$ be such that its entries form a $K(x)$-basis of A, and let $a \in K[x]$

and $B = (b_{i,j}) \in K[x]^{n\times n}$ be such that $aV' = BV$ and $\gcd(a, b_{1,1}, b_{1,2}, \dots, b_{n,n}) = 1$. Let $p = (p_1, \dots, p_n) \in K[x]^n$. Then

$$(pV)' = \sum_{i=1}^{n} (p_i \nu_i)' = \frac{ap' + pB}{a} V. \quad (4)$$

This motivates us to introduce the following definition.

Definition 10. *Let the map $\phi_V\colon K[x]^n \to K[x]^n$ be defined by $\phi_V(p) = ap' + pB$ for any $p \in K[x]^n$. We call ϕ_V the* map for polynomial reduction *with respect to V, and call the subspace* $\operatorname{im}(\phi_V) = \{\phi_V(p) \mid p \in K[x]^n\}$ *the* subspace for polynomial reduction *with respect to V.*

Note that, by construction and because of Lemma 9, $q \in K[x]^n$ is in $\operatorname{im}(\phi_V)$ if and only if $\frac{q}{a}V$ is integrable in A.

We can always view an element of $K[x]^n$ (resp. $K[x]^{n\times n}$) as a polynomial in x with coefficients in K^n (resp. $K^{n\times n}$). In this sense we use the notation $\operatorname{lc}(\cdot)$ for the leading coefficient and $\operatorname{lt}(\cdot)$ for the leading term of a vector (resp. matrix). For example, if $p \in K[x]^n$ is of the form

$$p = p^{(r)}x^r + \cdots + p^{(1)}x + p^{(0)}, \quad p^{(i)} \in K^n,$$

then $\deg_x(p) = r$, $\operatorname{lc}(p) = p^{(r)}$, and $\operatorname{lt}(p) = p^{(r)}x^r$. Let $\{e_1, \dots, e_n\}$ be the standard basis of K^n. Then the module $K[x]^n$ viewed as a K-vector space is generated by

$$\mathcal{S} := \{e_i x^j \mid 1 \le i \le n,\, j \in \mathbb{N}\}.$$

We define $K[x]^n_\mu := \{p \in K[x]^n \mid \deg_x(p) \le \mu\}$; as a K-vector space it is generated by

$$\mathcal{S}_\mu := \{e_i x^j \mid 1 \le i \le n,\, 0 \le j \le \mu\}.$$

Any element $p \in K[x]^n_\mu$ can be expressed in the basis $\mathcal{S}_\mu$ as a vector $\vec{p} \in K^{n(\mu+1)}$ (in the following the decoration $\vec{\ }$ always indicates such a typecast).

Definition 11. *Let N_V be the K-subspace of $K[x]^n$ generated by*

$$\{t \in \mathcal{S} \mid t \ne \operatorname{lt}(p) \text{ for all } p \in \operatorname{im}(\phi_V)\}.$$

Then $K[x]^n = \operatorname{im}(\phi_V) \oplus N_V$. We call N_V the standard complement *of* $\operatorname{im}(\phi_V)$. *For any $p \in K[x]^n$, there exist $p_1 \in K[x]^n$ and $p_2 \in N_V$ such that*

$$\frac{p}{a}V = (p_1V)' + \frac{p_2}{a}V.$$

This decomposition is called the polynomial reduction *of p with respect to V.*

Proposition 12. *Let $a \in K[x]$ and $B \in K[x]^{n\times n}$ be such that $aV' = BV$, as before. If $\deg_x(B) \le \deg_x(a) - 1$, then N_V is a finite-dimensional K-vector space.*

Proof. In addition to the proof of the assertion, we also explain how to determine the dimension and a basis for N_V, for later use. For brevity, let $\mu := \deg_x(a) - 1$. We distinguish two cases.

Case 1. Assume that $\deg_x(B) < \mu$. For any $p \in K[x]^n$ of degree $s > 0$, we have

$$\operatorname{lt}(\phi_V(p)) = s\operatorname{lc}(a)\operatorname{lc}(p)x^{s+\mu}.$$

Thus all monomials $e_i x^j \in \mathcal{S}$ with $1 \le i \le n$ and $j \ge \mu + 1$ are not in N_V. Let $\vec{B}_1, \dots, \vec{B}_n$ be the columns of B, expressed in the basis $\mathcal{S}_\mu$. Let $C(B)$ be the K-subspace of $K[x]^n_\mu$ generated by these column vectors. If $q \in \operatorname{im}(\phi_V)$, then $q = \phi_V(p) = pB$ for some $p \in K^n$, which implies that $\vec{q}$ is a linear combination of $\vec{B}_i$'s. Then $K[x]^n_\mu = C(B) \oplus N_V$. So $\dim_K(N_V) = (\mu + 1)n - \dim_K(C(B))$ and a basis of N_V can be computed by looking at the echelon form of the matrix $(\vec{B}_1, \dots, \vec{B}_n)$.

Case 2. Assume that $\deg_x(B) = \mu$. For any $p \in K[x]^n$ of degree s, we have

$$\operatorname{lt}(\phi_V(p)) = \operatorname{lc}(p)(s\operatorname{lc}(a)I_n + \operatorname{lc}(B))x^{s+\mu}.$$

Let ℓ be the largest nonnegative integer such that $-\ell\operatorname{lc}(a)$ is an eigenvalue of $\operatorname{lc}(B) \in K^{n\times n}$. Then for any $s > \ell$, the matrix $J_s = s\operatorname{lc}(a)I_n + \operatorname{lc}(B)$ is invertible. So any monomial $e_i x^j$ with $j > \ell + \mu$ is not in N_V for any $i = 1, \dots, n$. Let $p = \sum_{i=1}^n \sum_{j=0}^{\ell} p_{i,j} e_i x^j$. Then $\phi_V(p)$ belongs to $K[x]^n_{\ell+\mu}$. In the basis $\mathcal{S}_{\ell+\mu}$, we can express $\phi_V(p)$ as a vector of length $n(\ell + \mu + 1)$ with entries linear in the $p_{i,j}$'s. This vector can be written in the form $M_\ell \vec{P}$, where $\vec{P} = (p_{1,0}, p_{2,0}, \dots, p_{n,\ell})^T$ and $M_\ell \in K^{n(\ell+\mu+1)\times n(\ell+1)}$. Every $q \in K[x]^n_{\ell+\mu}$ can be expressed as a vector $\vec{q} \in K^{n(\ell+\mu+1)}$. Then $q \in \operatorname{im}(\phi_v)$ if and only if $\vec{q}$ is in the column space of M_ℓ. Therefore,

$$K[x]^n_{\ell+\mu} = C(M_\ell) \oplus N_V.$$

This implies that $\dim_K(N_V) = n(\ell+\mu+1) - \operatorname{rank}(M_\ell)$, and a basis of N_V can be computed by looking at the echelon form of the matrix M_ℓ. ∎

In general, the condition $\deg_x(B) \le \deg_x(a) - 1$ may not hold for an arbitrary basis V of A. The following lemma shows that we can perform a simple change of basis to make the condition hold.

Lemma 13. *Let $W = \{\omega_1, \dots, \omega_n\}$ be an integral basis of A such that it is also normal at infinity. Then there exist nonnegative integers $\tau_1, \dots, \tau_n$ such that*

$$V := \{\nu_1, \dots, \nu_n\} \quad \text{with } \nu_i = x^{-\tau_i}\omega_i$$

is a basis of A which is normal at 0 and integral at all other places (including infinity).

Proof. It is clear that such a basis V will be normal at zero, because multiplying the generators by the rational functions x^{τ_i} brings it back to a global integral basis, which is in particular a local integral basis at zero. It is also clear that such a basis will be integral at every other point $a \in \bar{K} \setminus \{0\}$, because the multipliers $x^{-\tau_i}$ are locally units at such a. Finally, since the original basis is normal at infinity, there exist rational functions $u_1, \dots, u_n$ such that $\{u_1\omega_1, \dots, u_n\omega_n\}$ is a local integral basis at infinity. Since u_i can be written as $u_i = x^{-\tau_i}\tilde{u}_i$ with $\tau_i \in \mathbb{Z}$ and $\tilde{u}_i$ being a unit in $\bar{C}(x)_\infty$, we see that also V is a local integral basis at infinity. The integers τ_i can only be nonnegative because the ω_i's have no finite poles and therefore each of them is either constant or has a pole at infinity by Chevalley's theorem. ∎

Combining the Hermite reduction and polynomial reduction, we get the following theorem.

Theorem 14. *Let W be an integral basis of A that is normal at infinity. Let $T := \operatorname{diag}(x^{-\tau_1}, \dots, x^{-\tau_n}) \in K(x)^{n\times n}$ be such that $V = TW$ is integral at infinity. Let $e \in K[x]$,*

$\lambda \in \mathbb{N}$, *and* $B, M \in K[x]^{n\times n}$ *be such that* $eW' = MW$ *and* $x^\lambda eV' = BV$. *Then any element* $f \in A$ *can be decomposed into*

$$f = g' + \frac{1}{d}PW + \frac{1}{x^\lambda e}QV, \tag{5}$$

where $g \in A$, $d \in K[x]$ *is squarefree and* $\gcd(d, e) = 1$, $P, Q \in K[x]^n$ *with* $\deg_x(P) < \deg_x(d)$ *and* $Q \in N_V$, *which is a finite-dimensional* K*-vector space. Moreover,* P, Q *are zero if and only if* f *is integrable in* A.

Proof. After performing the Hermite reduction on f, we get

$$f = \tilde{g}' + \frac{1}{d}PW + \frac{1}{e}UW,$$

where $P = (v_1, \ldots, v_n) \in K[x]^n$ and $U = (u_1, \ldots, u_n) \in K[x]^n$ with u_i, v_i introduced in (3). By Lemma 13, there exists $T := \operatorname{diag}(x^{-\tau_1}, \ldots, x^{-\tau_n}) \in K(x)^{n\times n}$ such that $V = TW$ is integral at infinity. By the same lemma it follows that V is also normal at 0 and integral at all other places. Note that we can choose T as the identity matrix if $\deg_x(M) \le \deg_x(e) - 1$. By taking derivatives, we get

$$V' = \left(T' + T\frac{M}{e}\right)T^{-1}V = \frac{B}{a}V,$$

where $a = x^\lambda e$ for some $\lambda \in \mathbb{N}$ and $B \in K[x]^{n\times n}$. Since V is locally integral at infinity, $\deg_x(B) \le \deg_x(a) - 1$ by Lemma 4. By expanding in terms of the new basis V, we get

$$\frac{1}{e}UW = \frac{1}{a}\tilde{U}V,$$

where $\tilde{U} = x^\lambda UT^{-1} \in K[x]^n$. Next, we decompose $\tilde{U}$ into $\tilde{U} = \phi_V(\tilde{U}_1) + \tilde{U}_2$ with $\tilde{U}_1, \tilde{U}_2 \in K[x]^n$ and $\tilde{U}_2 \in N_V$. Then we get

$$\frac{1}{e}UW = (\tilde{U}_1V)' + \frac{1}{a}\tilde{U}_2V.$$

We then get the decomposition (5) by setting $g = \tilde{g} + \tilde{U}_1V$ and $Q = U_2$.

Assume that f is integrable. Then Lemma 9 implies that $d \in K$. Since $\deg_x(P) < \deg_x(d)$, we have $P = 0$. Then

$$\frac{1}{x^\lambda e}QV = \sum_{i=1}^{n}(a_i\nu_i)'$$

for some $a_i \in K[x]$. So $Q \in \operatorname{im}(\phi_V)$. Since $\operatorname{im}(\phi_V) \cap N_V = \{0\}$, it follows that $Q = 0$. ∎

The decomposition in (5) is called an *additive decomposition* of f with respect to x.

6. TELESCOPING VIA REDUCTIONS: SECOND APPROACH

We now discuss how to compute telescopers for algebraic functions via Hermite reduction and polynomial reduction.

Let W, V, e, λ, M, B be as in Theorem 14. To construct a telescoper for $f \in A$, we first consider the additive decompositions of the successive derivatives $\partial_t^i \cdot f$ for $i \in \mathbb{N}$. Assume that

$$\partial_t \cdot W = \frac{1}{\tilde{e}}\tilde{M}W \quad \text{and} \quad \partial_t \cdot V = \frac{1}{x^{\tilde{\lambda}}\tilde{e}}\tilde{B}V,$$

where $\tilde{e} \in K[x]$, $\tilde{M}, \tilde{B} \in K[x]^{n\times n}$, and $\tilde{\lambda} \in \mathbb{N}$. Since ∂_t and ∂_x commute, Proposition 7 in [8] implies that $\tilde{e} \mid e$ and $x^{\tilde{\lambda}}\tilde{e} \mid x^\lambda e$, as polynomials in $K[x]$. So we can just take $\tilde{e} = e$ and $\tilde{\lambda} = \lambda$ by multiplying $\tilde{M}, \tilde{B}$ by some factors of $x^\lambda e$. A direct calculation yields $\partial_t \cdot f = (\partial_t \cdot g)' + h$, where

$$h = \left(\partial_t \cdot \frac{P}{d} + \frac{P\tilde{M}}{de}\right)W + \left(\partial_t \cdot \frac{Q}{x^\lambda e} + \frac{Q\tilde{B}}{x^{2\lambda}e^2}\right)V.$$

This implies that the squarefree part of the denominator of h divides xde. Applying Hermite reduction and polynomial reduction to h yields

$$h = \tilde{g}_1' + \frac{1}{d}P_1W + \frac{1}{x^\lambda e}Q_1V,$$

where $P_1, Q_1 \in K[x]^n$ with $\deg_x(P_1) < \deg_x(d)$ and $Q_1 \in N_V$. Repeating this discussion, we get the following lemma.

Lemma 15. *For any* $i \in \mathbb{N}$, *the derivative* $\partial_t^i \cdot f$ *has an additive decomposition of the form*

$$\partial_t^i \cdot f = g_i' + \frac{1}{d}P_iW + \frac{1}{x^\lambda e}Q_iV,$$

where $g_i \in A$, $P_i, Q_i \in K[x]^n$ *with* $\deg_x(P_i) < \deg_x(d)$ *and* $Q_i \in N_V$.

As application of the above lemma, we can compute the minimal telescoper for f by finding the first linear dependence among the (P_i, Q_i) over K. We also obtain an upper bound for the order of telescopers.

Corollary 16. *Every* $f \in A$ *has a telescoper of order at most* $n\deg_x(d) + \dim_K(N_V)$.

Example 17. *We continue with Example 8, by applying the polynomial reduction to the Hermite remainders* h_0, h_1, h_2. *The matrix* M *computed before satisfies the degree condition of Proposition 12, so no change of basis is needed. First we compute polynomials* $u_i, v_i \in K[x, y]$ *such that for* $i = 0, 1, 2$ *we have*

$$h_i = \frac{u_i}{e} + \frac{v_i}{d} = \frac{u_i}{27x^2 + 54xt + 27t^2 + 4} + \frac{v_i}{x}.$$

By noting that $\deg_x(u_i) = 1$ *and* $\deg_x(e) = 2$, *we see that the map for the polynomial reduction* $\phi(p) = ep' + pM$ *can only be applied for* $p \in K^n$ *so that it turns into* $\phi(p) = pM$. *This means that we reduce* xy^2 *using the third row of* M *and* xy *using its second row. A straightforward calculation reveals that* h_0, h_1, *and* h_2 *all reduce to* 0. *Hence we are left with finding a* K*-linear combination among the* v_i*:*

$$v_0 = \frac{-6y^2 + 9yt - 4}{27t^2 + 4},$$

$$v_1 = \frac{6\big(27y^2t - (27t^2 - 2)y + 18t\big)}{(27t^2 + 4)^2},$$

$$v_2 = \frac{6\big((96 - 1539t^2)y^2 + (1215t^3 - 306t)y - 1026t^2 + 64\big)}{(27t^2 + 4)^3}.$$

As expected, we obtain the same telescoper as in Example 8.

7. THE D-FINITE CASE

With algebraic functions being settled, it is natural to wonder about a possible reduction-based creative telescoping algorithm for D-finite functions. Recall that in this setting we consider an operator $L \in K(x)[\partial_x]$ instead of a minimal polynomial $m \in K[x, y]$ and instead of an algebraic field extension $K(x)[y]/\langle m\rangle$ we consider the $K(x)[\partial_x]$-left-module $A = K(x)[\partial_x]/\langle L\rangle$. Then the element $1 \in A$ is a solution of L because $L \cdot 1 = L = 0$ in A by construction. If $n = \deg_{\partial_x} L$, then the general element of A has the form $f = f_0 + f_1\partial_x + \cdots + f_{n-1}\partial_x^{n-1}$ for some $f_0, \ldots, f_{n-1} \in K(x)$. Very much as in the algebraic case, there is a natural way to associate certain series objects to the elements of A. Based on these series objects, a notion of integrality was proposed last year [17], and an algorithm for computing integral bases has been given for so-called Fuchsian operators L.

It turns out that the Hermite reduction of Section 3 also works in this setting, if we say that a term $(x-a)^\alpha \log(x)^\beta$ in a generalized series solution is integral if and only if $\alpha \geq 0$. Note that then $\log(x)$ will be considered integral at zero, despite the singularity of the complex function at this point. This has the somewhat counterintuitive consequence that $\log(x)$ is integral at every $a \in \bar{K} \cup \{\infty\}$ although it does not have a pole anywhere. For algebraic functions, this is not possible by Chevalley's theorem, and this fact enters in an essential way in the proofs of Sections 4 and 5. The lack of Chevalley's theorem is not an artefact of a (possibly wrong) treatment of logarithmic terms. Because of the Fuchs relation [23, p. 241] there exist operators $L \in K(x)[\partial_x]$ whose series solutions at any point $a \in \bar{K} \cup \{\infty\}$ have no logarithmic terms, only nonnegative exponents, and which are nevertheless not constant.

For the time being, the existence of such operators is a severe obstruction to a possible generalization of the termination arguments for reduction-based creative telescoping from algebraic functions to Fuchsian D-finite functions. We hope to explore this topic further in the future.

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