AN ALGEBRAIC STUDY OF LINKAGES WITH HELICAL JOINTS

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Abstract. Methods from algebra and algebraic geometry have been used in various ways to study linkages in kinematics. These methods have failed so far for the study of linkages with helical joints (joints with screw motion), because of the presence of some non-algebraic relations. In this article, we explore a delicate reduction of some analytic equations in kinematics to algebraic questions via a theorem of Ax. As an application, we give a classification of mobile closed 5-linkages with revolute, prismatic, and helical joints.

Contents

1. Introduction

Linkages, and in particular closed linkages, are a crucial object of study in the modern theory of kinematics. The use of algebra and geometry for studying linkages is very classical and goes back to Sylvester, Kempe, Cayley and Chebyshev.

A linkage, as appearing in robotics/mechanical engineering, biology, as well as modelling of molecules in chemistry, etc., is a mechanical structure that consists of a finite number of rigid bodies – its *links* – and a finite number of *joints* that connect the links together, so that they possibly produce a motion. A linkage is called *closed* if its number of links and joints are equal and they are connected cyclically. We consider four types of joints:

(R) revolute joints: allow rotations around a fixed axes;
(P) prismatic joints: allow translations in a fixed direction;
(C) cylindrical joints: allow rotations around a fixed axes and translations in the the direction of the axes;
(H) helical joints: allow the motions of a cylindrical joint where the rotation angle and the translation length are coupled by a linear equation.

We will use the notation R-joint for a revolute joint, and similarly for other types. Note that the dimension of the set of allowed motions (the degree of freedom) is 1 for joints of type R, P, and H, and it is 2 for C-joints.

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The *configuration set* of a closed linkage L, denoted by K_L, is the set of possible simultaneous motions of all joints (see Definition 1 for a precise description). The dimension of K_L is called the *mobility* of L, and L is called *mobile* if the mobility is positive.

It is known that K_L can be described by analytic equations; see [14], page 356. If there are no H-joints, then we also have a description by algebraic equations. We refer to [13] for a historic overview of the use of geometric algebra in kinematics. This is the subject that has attracted algebraists the most. It should be mentioned, however, that also number theory has been used for studying linkages: in [9], reduction modulo prime numbers are considered in order to construct a new family of Stewart-Gough platforms. Below, in Section 2 we briefly explain the algebraic setup, as well as the theory of bonds, a rather new combinatorial technique that has shown to be very useful for analysing closed linkages with R-joints [10]. We also explain the analytical relations in the presence of H-joints. There are also other (numerical) algebraic methods that are applied in kinematics, see for example [3, 7, 15].

A closed linkage with n joints, where all joints are R-joints, is denoted by nR-linkage. We denote by n-linkage a linkage with n joints where no information on the type of joints is specified. It is easy to imagine that a 3R-linkage does not have a motion, and hence its configuration set is trivial. On the other hand a generic nR-linkage for $n \geq 7$ has positive mobility (see [14], page 356), and hence there is not much to study. So the interesting cases are when $n = 4, 5$ or 6. Nowadays we have a full classification of 4R- and 5R-linkages with mobility one ($\dim K_L = 1$), and we know many cases for $n = 6$ (see [8]). It is an open research problem to classify all 6R-linkages. These classification problems are considered by algebraists. Because of the nature of other types of linkages it seemed difficult, or rather impossible, to be able to use any of the present algebraic techniques for linkages with H-joints.

What is new in this article? As a main result, we show that unexpected mobility of a linkage with H-joints, i.e., a mobility that is strictly bigger than predicted by the Chebyshev-Grübler-Kutzbach formula which simply counts parameters and equational restrictions, can always be explained algebraically. Let L be a linkage with H-joints, and let L' be the linkage obtained from L by replacing all H-joints by C-joints. It is clear that the configuration set of L is a subset of the configuration set of L'. The relation between the configuration set of L' and the Zariski closure of the configuration set of L will be made very precise (Theorem 9), with the help of Ax's theorem [1] on the transcendence degree of function fields with exponentials. Note that Ax's theorem is originally about Schanuel's conjecture in number theory and has no apparent connections to kinematics.

The mobile linkages with 4 joints of type H, R, or P have been classified in [6]. Here (more precisely in Theorem 10) we give a classification of mobile linkages with 5 joints of type H, R, or P. Using our main result, we reduce to linkages with joints of type R or P only. The classification of mobile 5R-linkages has been done in [12], but for linkages with both R-joints and P-joints, we could not find a complete classification in the literature. On the other hand, this classification is not difficult when we use the theory of bonds, so we also give it in here (Theorem 6).

Structure of the paper. In Section 2 we set up a mathematical language to describe and to analyse linkages with arbitrary types of joints, we recall the theory of bonds for R-joints and we introduce the adaptations of this theory that make it work also for P-joints. In Section 3 we use these algebraic methods to classify mobility for 5-linkages with P- and R-joints. In Section 4 we reduce H-joints to C-joints. Finally, in Section 5 we use Ax's theorem and the results of the previous sections to classify mobile 5-linkages with helical joints.

Because the two proofs of the classification results, i.e., Theorem 6 and Theorem 10, are methodically quite different, it is possible to apply a "filter" while reading in case one is only interested in the main result (or in one of the proofs using Ax's theorem). In that case, the reader could omit the second half of Section 2, right before bonds are introduced, and the whole Section 3, except for Theorem 6 which has to be taken for granted.

2. Algebraic setup

In this section we set up the notation for an algebraic description of linkages with arbitrary joints. Then we briefly recall bonds for R-joints as defined in [10]. Finally, we introduce bonds for P-joints and prove some basic properties of them.

Dual quaternions and configuration set. Suppose $\mathbb{R}$ is the set or real numbers, $\mathbb{D} := \mathbb{R} + \epsilon\mathbb{R}$ is the ring of dual numbers and $\epsilon^2 = 0$. Denote by $\mathbb{H}$ the non-commutative algebra of quaternions, where

$$\mathbb{H} = \{A = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \text{ where } \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 \text{ and } \mathbf{i}.\mathbf{j} = \mathbf{k}\,, \mathbf{j}.\mathbf{k} = \mathbf{i}\,, \mathbf{k}.\mathbf{i} = \mathbf{j}\}.$$

Let $\mathbb{DH} := \mathbb{D} \otimes_{\mathbb{R}} \mathbb{H}$ denote the dual quaternions, i.e.

$$\mathbb{DH} = \{h = A + \epsilon B \text{ where } A, B \in \mathbb{H} \text{ and } \epsilon^2 = 0\}.$$

We call A the *primal* part of a dual quaternion h, and B the *dual* part of h. The conjugate of $A \in \mathbb{H}$, as above, is defined by $\overline{A} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$. This extends naturally to define the conjugate dual quaternion of h by

$$\overline{h} = \overline{A} + \epsilon\overline{B}$$

The norm function $\mathrm{N}\colon \mathbb{DH} \to \mathbb{D}$ is then defined by $\mathrm{N}(h) = h.\overline{h} = A\overline{A} + \epsilon(A\overline{B} + \overline{A}B)$; and the latter is called the *norm* of h.

Note that $\mathbb{DH}$ can be regarded as a real 8-dimensional vector space, and projectivising $\mathbb{DH}$ we obtain $\mathbb{P}^7$. The *Study quadric* S is a hypersurface of this projective space defined by the quadratic equation

$$\sum_{i=0}^{3} a_i b_i = 0$$

where $h = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} + \epsilon(b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$. In other words

$$S = \{h \in \mathbb{P}^7 \text{ such that } \mathrm{N}(h) \in \mathbb{R}\}$$

The linear 3-space represented by all dual quaternions with zero primal part is denoted by E. It is contained in the Study quadric, and the complement $S - E$ is closed under multiplication and multiplicative inverse; hence $S - E$ forms a group, which is isomorphic to the group of Euclidean displacements (see [11, Section 2.4]).

For a natural number $n \in \mathbb{N}$, a *linkage* with n joints is described as an n-tuple $L = (j_1, \ldots, j_n)$, where each j_i represents a joint. We will use cyclic notation for joint indices, i.e., $j_{n+1} = j_1$.

Quantisation. The type of joint specifies which data must be given in order to determine the set of possible motions, as follows. Suppose $k \in \{1, \ldots, n\}$.

I. If j_k is an *R-joint*, then we specify a dual quaternion h_k such that $h_k^2 = -1$. We write p_k and q_k for the primal and the dual part of h_k, i.e., $h_k = p_k + \epsilon q_k$ with $p_k, q_k \in \mathbb{H}$. The set of possible motions is parameterised by the joint parameter $t_k \in \mathbb{P}^1$, which is the rotation with an angle of $2\,\mathrm{arccot}(t_k)$. The rotation corresponds to the dual quaternion $m_k = t_k - h_k$, and to 1 if $t_k = \infty$. Note that the latter means we have fixed the initial position at ∞.

II. If j_k is a P-joint, then we specify a quaternion $p_k \in \mathbb{H}$ such that $p_k^2 = -1$. The set of possible motions is parameterised by the joint parameter $s_k \in \mathbb{R}$, and the translation corresponds to the dual quaternion $m_k = 1 - \epsilon s_k p_k$.

III. If j_k is a *C-joint*, then we specify a dual quaternion $h_k = p_k + \epsilon q_k$ such that $h_k^2 = -1$, The set of possible motions is parameterised by the joint parameters (s_k, t_k) and corresponds to the dual quaternion $m_k = (1 - \epsilon s_k p_k)(t_k - h_k)$.

IV. If j_k is an *H-joint*, then we specify a dual quaternion $h_k = p_k + \epsilon q_k$ such that $h_k^2 = -1$, and a nonzero real number g_k. The number $\frac{g_k}{2\pi}$ is often refereed to as the *pitch* in mechanical engineering. The joint parameter is $\alpha_k \in \mathbb{R}$, and the motion corresponds to the dual quaternion $m_k = (1 - \epsilon g_k \alpha_k p_k)(1 - \tan(\frac{\alpha_k}{2})h_k)$.

The data which must be specified for all joints are called the *geometric parameters*. Note that when the linkage moves, the geometric parameters also change. However, there are functions in the geometric parameters that do not change when the linkage moves, such as the normal distance and the angle between neighbour rotation or helical axes.

Definition 1. The *configuration set* K is the set of all parameters t_k, s_k, α_k such that the closure equation

$$m_1 m_2 \cdots m_n \equiv 1 \tag{1}$$

is fulfilled. The symbol $\equiv$ stands for projective equivalence, i.e., up to multiplication by a nonzero real scalar.

The *mobility of* K is the dimension of the solution set of Equation 1 as a complex analytic set in the parameter space. If it is positive, then we say that L is *mobile*.

We are interested in mobile linkages, and mainly those with mobility 1. Finding such linkage for given types, and numbers, of joints is a main goal. This leads to analysing the solutions of Equation (1).

If all values for t-parameters are ∞ and all values for s- and α-parameters are 0, then all m_k are equal to 1 and Equation (1) is fulfilled. This point of K is called the *initial configuration* of L.

Remark 1. The dimension of K as a real analytic set would be a more interesting number than the mobility we defined above, but it is harder to control. For instance, planar 4R-linkages always have mobility 1, but the real dimension can also be 0. In any case, the complex dimension is an upper bound for the real dimension, and if the two numbers are not equal then all real configurations must be singularities of the complex configuration space.

The remaining part of this section is concerned with the theory of bonds. If the reader is willing to believe Theorem 6, he/she may jump forward to this theorem, skip its proof and proceed with Section 4, in order to get faster to the application of Ax's theorem.

Bonds. For linkages with R-joints, bonds have been introduced in [10] as an algebraic tool which is used to describe and understand the algebraic structure of the configuration set. Informally speaking, bonds are the points in the boundary of the compactification of the complex configuration set. The closure equation degenerates in the boundary, and one obtains useful algebraic consequences. At this moment, we do not have any geometric intuition for bonds, we just use them mainly as a tool for studying the (real) configuration set and for deriving geometric conditions of the rotation axes. Here is the precise definition.

Definition 2. Let L be an n-linkage with joints of type R or P. Suppose Z is the projective closure of the complexification of K_L in $(\mathbb{P}^1_{\mathbb{C}})^n$. Then the *bond set* B is defined as the intersection of Z and the solution set of the bond equation

$$m_1 m_2 \cdots m_n = 0. \tag{2}$$

Remark 2. In [10], where the theory of bonds is initially developed, only linkages with mobility 1 are considered and bonds are defined as points on the normalisation of the curve K. In this paper, however, this is not necessary because we do not need multiplicities of bonds. Hence we can afford to simply say a bond is a point of B.

Construction of the bond diagram. Let β be a bond. We say that β is *attached* to a joint m_k if $\mathrm{N}(m_k(\beta)) = 0$. This is equivalent to $t_k^2 + 1 = 0$ if j_k is an R-joint, and to $s_k = \infty$ if j_k is a P-joint. If β is attached to two different joints j_k and j_ℓ, then we say that β *connects* j_k and j_ℓ if and only if

$$m_k(\beta)m_{k+1}(\beta)\cdots m_\ell(\beta) = m_\ell(\beta)m_{\ell+1}(\beta)\cdots m_k(\beta) = 0$$

(this definition is slightly different from the definition in [10], but the more complicated definition using multiplicities is not needed here).

Definition 3. Suppose $G = (V, E)$ is the graph of a linkage, where vertices, elements of V, represent the rigid bodies of L and $v_i, v_i \in V$ are connected via an edge $e \in E$ is there is a joint between them. The *bond diagram* is then defined to be this graph together with the following extra information: two edges are connected if their corresponding joints are connected via a bond.

Example 1. There is a unique family of 4R linkages such that the four axes are not all parallel and do not all have a common point, the Bennett linkage (see [4]). Its configuration curve can be defined by the equations

$$t_1 = t_3, t_2 = t_4, at_1s - t_2 + b = 0,$$

where $a, b \in \mathbb{R}$, $(a, b) \neq (1, 0)$, $a \neq 0$ are parameters. The bond set is

$$B = \{(\pm \mathrm{i}, a \pm \mathrm{i} + b, \pm \mathrm{i}, a \pm \mathrm{i} + b), ((\pm \mathrm{i} - b)/a, \pm \mathrm{i}, (\pm \mathrm{i} - b)/a, \pm \mathrm{i})\},$$

and the bond diagram is shown in Figure 1.

We define the offset $o(h_1, h_2, h_3)$ of three lines h_1, h_2, h_3 as follows. We assume that neither $h_1||h_2$ nor $h_2||h_3$, where the symbol $||$ is used to show the two lines are parallel (otherwise, the offset is not defined). Let n_{12} be the common normal of h_1, h_2, i.e., the unique line intersecting both h_1 and h_2 at a right angle. Let n_{23} be the common normal of h_2, h_3. Then $o(h_1, h_2, h_3)$ is defined as the signed distance between the intersection of h_2, n_{12} and the intersection of h_2, n_{23}. The sign comes from the orientation of the line h_2 represented as a dual quaternion. The offset of three consecutive R-joints is fixed when the linkage moves.

Example 2. Assume that the lines h_1, h_2, h_3 are coplanar and pairwise not parallel. Then $o(h_1, h_2, h_3)$ is the distance of the intersection points $h_1 \cap h_2$ and $h_2 \cap h_3$. If h_3 is rotated around h_2 by an angle different from π, call the result h_3', then h_1, h_2, h_3' will not be coplanar, but we still have $o(h_1, h_2, h_3) = o(h_1, h_2, h_3')$.

We recall some well-known facts on the bond diagram, and refer to [10] for details.

(i) Every bond is attached to at least two joints.
(ii) If a bond is attached to a joint j_k, then it connects j_k to at least one other joint.
(iii) If a joint j_k actually moves during the motion, then it is attached to at least one bond.
(iv) Two consecutive R-joints, j_i and j_{i+1}, are not connected by a bond.
(v) If j_i, j_{i+1} and j_{i+2} are R-joints with axes h_i, h_{i+1} and h_{i+2} such that j_i is connected to j_{i+2} and $h_1||h_2$, then $h_2||h_3$.
(vi) If j_i, j_{i+1} and j_{i+2} are R-joints with axes h_i, h_{i+1} and h_{i+2} such that h_i is not parallel to h_{i+1}, and j_i is connected to j_{i+2}, then $o(h_i, h_{i+1}, h_{i+2}) = 0$.

3. 5-linkages with Revolute and Prismatic Joints

In this section we classify mobile closed 5-linkages with R- and P-joints. For the case of R-joints only, this is well-known, as described below. The general case is handled by bond theory; this makes the proof quite conceptual and avoids long and technical calculations. We use the results of this section to classify 5-linkages with helical joints in Section 5.

If L has two neighbouring R-joints with equal axes or two neighbouring P-joints with equal directions, then we say that L is *degenerate*. Throughout this section, we assume that $n = 5$ or $n = 4$, and $L = (j_1, \ldots, j_n)$ is a mobile linkage with configuration set K. We also assume that L is not degenerate, and that no joint parameters are constant during motion of the linkage (otherwise one could easily make n smaller).

If L has only R-joints, then we have one of the following three cases ([12]; see [10] for a proof using bond theory).

(1) L is spherical, i.e., all rotation axes meet in the same point; then L has mobility 2.
(2) L is planar, i.e., all rotation axes are parallel; then L has mobility 2.
(3) L is a Goldberg linkage, constructed as follows: take two spatial 4-linkages with one joint and one link in common; then remove the common link. The mobility of the Goldberg linkage is 1. If $h_1, \ldots, h_5$ are the rotation axes, then

$$o(h_4, h_5, h_1) = o(h_5, h_1, h_2) = o(h_1, h_2, h_3) = 0, \text{ and } o(h_2, h_3, h_4) = \pm o(h_3, h_4, h_5)$$

up to cyclic permutation of joints.

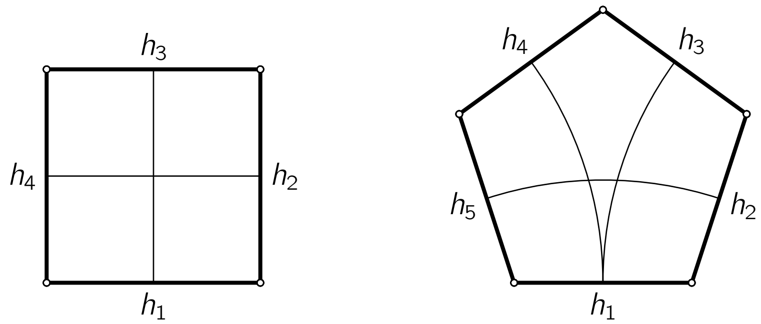

Figure 1. Bond diagrams for the Bennett 4R and the Goldberg 5R linkage. The vanishing of all offsets of the Bennett 4R linkage and of the three offsets $o(h_4, h_5, h_1)$, $o(h_5, h_1, h_2)$, and $o(h_1, h_2, h_3)$ of the Goldberg 5R linkage is an easy consequence of bond theory.

By considering the specified data and parameterised motions modulo ϵ, we may construct a spherical linkage L', the *spherical projection* of L. The P-joints of L disappear, their translation motions are projected to the identity. Parallel joint axes of L are projected to identical axes of L'. Note that K is projected to an algebraic subset of K' (the configuration set of L'). This subset has positive dimension if and only if L has at least one R-joint.

Lemma 3. *If L has 2 or more P-joints, then all rotation axes are parallel.*

Proof. Let $r \geq 2$ be the number of P-joints. Then L' is a spherical linkage with $5 - r \leq 3$ joints. Such a linkage is necessarily degenerate: if all three (or fewer) joints are actually moving, then all axes are identical. □

In order to classify PRRRR linkages, we use bond theory. Here are some additional facts for bonds in the presence of P-joints.

Lemma 4. *Assume that j_i is a P-joint, and j_{i+1} and j_{i+2} are R-joints.*
(a) The joints j_i and j_{i+1} cannot be connected by a bond.
(b) If the joints j_i and j_{i+2} are connected by a bond, then the axes h_{i+1} and h_{i+2} are parallel.

Proof. (a) Assume, without loss of generality, that (∞, t) are the coordinates at (j_i, j_{i+1}) of the bond connecting j_i and j_{i+1}. Then we have that $\epsilon p_i(t - h_{i+1}) = 0$, which is impossible.

(b) If (∞, t, t') are the coordinates at (j_i, j_{i+1}, j_{i+2}) of the bond connecting j_i and j_{i+1}, then $\epsilon p_i(t - h_{i+1})(t' - h_{i+2}) = 0$. We pass to the dual part. Since p_i is invertible, we conclude that $(t - p_{i+1})(t' - p_{i+2}) = 0$. This is only possible if $t^2 + 1 = t'^2 + 1 = 0$ and $p_{i+1} = \pm p_{i+2}$. □

Lemma 5. *If j_1 is a P-joint and all other joints are R-joints, then $h_2 || h_3$ and $h_4 || h_5$.*

Proof. Assume that j_1 is the P-joint. It must be connected by a bond to at least one other joint. By Lemma 4, this cannot be j_2 and j_5, so we may assume it is connected to j_3. By Lemma 4 again, h_2 and h_3 are parallel. Then the spherical projection L' is a 4-linkage with 2 equal axes p_2 and p_3. Hence L' is degenerate, and the other two axes p_4 and p_5 are equal too. □

Note that Lemma 5 holds for all j_i with appropriate indices.

Theorem 6. *Let L be a 5-linkage with at least one P-joint and all other joints of type R. Then the following two cases are possible.*

1. *Up to cyclic shift, j_1 is the only P-joint, $h_2 || h_3$, and $h_4 || h_5$; L has mobility 1, and $t_2 = \pm t_3$ and $t_4 = \pm t_5$ is fulfilled on the configuration curve.*
2. *All axes of R-joints are parallel.*

Proof. Using Lemma 5, we get $h_2 || h_3$, $h_4 || h_5$. Without loss of generality, we assume $p_2 = p_3$ and $p_4 = p_5$; if this is not true, it can be easily achieved by replacing h_3 by $-h_3$ or h_5 by $-h_5$. Either all axes of R-joints are parallel or the axes of h_3 and h_4 are not parallel. In the second case, there is nothing left to show; let us assume that h_3 and h_4 are not parallel. The primal part of the closure equation is equivalent to the equality of the two rotations

$$(t_2 - p_2)(t_3 - p_2) \equiv (t_5 + p_5)(t_4 + p_4).$$

Since the axes are distinct, both rotations must be the identity, which implies $t_2 = -t_3$ and $t_4 = -t_5$. □

4. Construction of Linkages with Helical Joints

In this section we give a construction that produces mobile linkages with H-joints from linkages with C-, P-, and R-joints. We illustrate the construction by several well-known examples and one example which is new.

We start with a simple construction: take a linkage with r C-joints that has mobility at least $r + 1$. For each C-joint j_k, impose the additional restriction $t_k = \cot(\frac{s_k}{2g_k})$ on its joint parameters (s_k, t_k), where g_k is a nonzero real constant. Any additional equation reduces the mobility at most by 1, so we get a mobile linkage where every C-joint j_k is replaced by an H-joint with pitch g_k.

We can extend this simple construction using the observation that $\mathbb{Q}$-linear relations between the angles imply algebraic relations between their tangents. For the general construction, which we call *screw carving*, we need the following ingredients.

1. a linkage L with m C-joints $j_{k_1}, \ldots, j_{k_m}$ and an undetermined number of R- and P-joints;
2. an irreducible analytic subspace K_0 of the configuration space of L;
3. an integer matrix A with m columns that annihilates the vector of analytic functions $(\alpha_{k_1}, \ldots, \alpha_{k_m})^t \in \mathbb{C}(K_0)^m$ such that $\cot(\frac{\alpha_k}{2}) = t_k$;
4. an m-tuple $(g_{k_1}, \ldots, g_{k_m})$ of nonzero real numbers, so that A also annihilates the vector of functions $(a_{k_1}, \ldots, a_{k_m})^t$, where $a_k : K_0 \to \mathbb{C}$ is the function $(s_*, t_*) \mapsto \frac{s_k}{g_k}$.

As before, the linkage L' with H-joints instead of C-joints is obtained by imposing the additional restriction $t_k = \cot(\frac{s_k}{2g_k})$ on its joint parameters (s_k, t_k), for each C-joint t_k. To obtain linkages with large mobility, the integer matrix A should have the largest possible rank, which means that all integral relations between the analytic angle functions are linear combination of matrix rows. (In the next section, we will indeed always choose such matrices of maximal rank.) The empty matrix with zero rows is allowed, then we just get the simple construction above.

Lemma 7. *Let $d := \dim(K_0)$ and $\ell := \operatorname{rank}(A)$. Then the mobility of the linkage produced by screw carving is at least $d - m + \ell$.*

Proof. The subset K' of K_0 that satisfies the additional restrictions $t_k = \cot(\frac{s_k}{2g_k})$ is contained in the configuration space of L'. Since the codimension of an analytic subset is never bigger than the number of defining equations, we see that $\dim(K') \geq d - m$. We claim that K' can be defined (as a subset of K') by only $m - \ell$ equations.

Let $\alpha_{k_1}, \dots, \alpha_{k_m} \in \mathbb{C}(K_0)$ be as above. The $\mathbb{Q}$-vector space generated by these m functions has dimension at most $m - \ell$. Without loss of generality, we assume that $\{\alpha_{k_1}, \dots, \alpha_{k_{m-\ell}}\}$ is a generating set. Any other α_k can be expressed as a $\mathbb{Q}$-linear combination

$$\alpha_k = q_1 \alpha_{k_1} + \cdots + q_{k_{m-\ell}} \alpha_{k_{m-\ell}},$$

with rational coefficients depending on the matrix A. But then we also have

$$\frac{s_k}{g_k} = q_1 \frac{s_{k_1}}{g_{k_1}} + \cdots + q_{k_{m-\ell}} \frac{s_{k_{m-\ell}}}{g_{k_{m-\ell}}}.$$

It follows that the equations $t_{k_1} = \cot(\frac{s_{k_1}}{2g_{k_1}}), \dots, t_{k_{m-\ell}} = \cot(\frac{s_{k_{m-\ell}}}{2g_{k_{m-\ell}}})$ imply all other equations. □

Example 3. Let L be a 4-linkage with 4 cylindrical joints with parallel axes. Its mobility is 4. For all configurations $(t_1 = \cot(\frac{\alpha_1}{2}), s_1, \dots, t_4 = \cot(\frac{\alpha_4}{2}), s_4)$, we have $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$ and $s_1 + s_2 + s_3 + s_4 = 0$. So we take K_0 as the full configuration set, A as the 1×4 matrix $(1, 1, 1, 1)$, and $g_1 = g_2 = g_3 = g_4$, and apply screw carving. We obtain a 4-linkage with 4 helical joints and mobility 4-4+1=1.

Similarly, one can obtain an n-linkage with n H-joints with parallel axes with mobility $n - 3$, $n \geq 4$.

Example 4. Here is a variation of the previous example. Set

$$h_1 = \mathbf{k} - \epsilon \mathbf{i}, h_2 = \mathbf{k} + \epsilon \mathbf{i}, h_3 = h_5 = \mathbf{k}, h_4 = \mathbf{k} + 2\epsilon \mathbf{j}$$

and let L be the CCRRR linkage with C-joint axes h_1, h_2 and R-joint axes h_3, h_4, h_5. Its mobility is 3, and all configurations satisfy $s_1 + s_2 = 0$. We define K_0 as the subvariety defined by $\tan(17 \operatorname{arccot}(t_1) - 11 \operatorname{arccot}(t_2)) = 0$ (this is a rational function in t_1, t_2). Its dimension is 2. We set as the 1×2 matrix $A = (1, 1)$ and $g_1 = \frac{1}{17}$, $g_2 = \frac{-1}{11}$. By screw carving we get an HHRRR linkage with mobility 1. Figure 2 shows the trace of the joint j_4 when the link with the two H-joints j_1, j_2 is fixed.

Example 5. Let h_1, h_2, h_3 be lines. Reflecting them by the coordinate axes represented by $\mathbf{i}$, we get $h_4 = \mathbf{i} h_1 \mathbf{i}$, $h_5 = \mathbf{i} h_2 \mathbf{i}$, $h_6 = \mathbf{i} h_3 \mathbf{i}$. Let L be the 6C-linkage with axes $h_1, \dots, h_6$. The zero set of the closure equation

$$(t_1 - h_1)(1 - \epsilon s_1 h_1) \cdots (t_6 - h_6)(1 - \epsilon s_6 h_6) \equiv 1$$

has a component of dimension 4, given by the equations

$$t_1 = t_4, t_2 = t_5, t_3 = t_6, s_1 = s_4, s_2 = s_5, s_3 = s_6, x\mathbf{i} + \mathbf{i}x = 0,$$

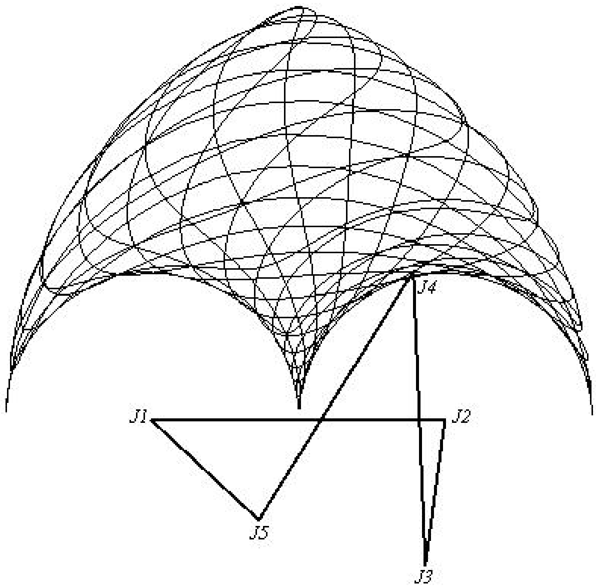

FIGURE 2. Planar projection of an HHRRR linkage with 5 parallel axes to the plane orthogonal to the axes (Example 4). The helical joints are at j_1 and j_2. The ratio of the pitches at the two helical joints j_1, j_2 is 11:17. The curve shown is the trace of the joint j_4. It is an algebraic curve of large degree.

where $x = (t_1 - h_1)(1 - \epsilon s_1 h_1)(t_2 - h_2)(1 - \epsilon s_2 h_2)(t_3 - h_3)(1 - \epsilon s_3 h_3)$.

With $A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$ and $g_1 = g_4$, $g_2 = g_5$, $g_3 = g_6$, the screw carving procedure gives a line symmetric 6H linkage with mobility 1.

Similarly, one can construct a plane symmetric RHHRHH linkage with mobility 1. Both linkages are well-known, see [2].

Example 6. Let h_1, h_2, h_3 be lines with linear independent primal parts that do not intersect pairwise, such that $o(h_1, h_2, h_3) = o(h_3, h_1, h_2) = 0$. Let L be the RRCRRC linkage with axes $h_1, h_2, h_3, h_2, h_1, h_3$. The zero set of the closure equation

$$(t_1 - h_1)(t_2 - h_2)(t_3 - h_3)(1 - \epsilon s_1 h_3)(t_4 - h_2)(t_5 - h_1)(t_6 - h_3)(1 - \epsilon s_6 h_3) \equiv 1$$

has two components of dimension 2. The first is given by $t_1 = -t_5$, $t_2 = -t_4$, $t_3 = t_6 = \infty$, $s_3 = s_6 = 0$; this is a degenerate motion which does not separate the pairs of axes at joints (j_1, j_5) and at (j_2, j_4). The equations of the second component K_0 can be computed by computer algebra. Two of them are $s_3 = s_6$ and $t_3 = t_6$; the remaining are more complicated. With $A = \begin{pmatrix} 1 & -1 \end{pmatrix}$ and $g_3 = g_6$, the screw carving procedure gives an RRHRRH linkage with mobility 1. In contrast to all families of mobile 6-linkages with H-joints that have been known up to now, this linkage has no parallel axes or apparent geometric symmetries. A distinctive property is the existence of a starting position with three pairs of coinciding axes.

5. Classification of 5-linkages with Helical Joints

Now we show that the construction in Section 4 is complete, i.e., every linkage with helical joints can be obtained in this way. The main idea is the application of a theorem by Ax to separate the transcendental part and the algebraic part of the closure equation. Then, we use the completeness result to classify 5-linkages with P-, R-, and H-linkages.

Let L be a mobile linkage with m helical joints; we assume that it is not degenerate, and that all joints actually move. There is a natural candidate for ingredients of the screw carving constructions in order to produce L.

(i) We define the *cylindrical extension* L' of L be replacing all H-joints by C-joints. The configuration set K of L can be naturally embedded in the configuration K' set of L'.
(ii) We take K_0 as the Zariski closure of K in K', i.e., the subset of K' defined by all algebraic equations that hold for K.
(iii) We take A as an integer matrix whose rows generate, as a $\mathbb{Z}$-module, the coefficient vectors of all integral linear equations that hold for the m α-parameters in K.
(iv) The m nonzero numbers are defined as the m pitches of K.

Application of the screw carving constructions just re-installs the screw conditions that already existed in L. But our construction includes a prediction on the mobility, and it is not clear if the mobility of L' is big enough to explain the mobility of L.

Since the use of number theoretic theorems is not usual in kinematics, we include the full statement of the Ax theorem on the transcendence degrees of a function field with exponential functions.

Theorem 8. *Let q, n be positive integers. Let $f_1, \dots, f_q$ be analytic functions in some neighbourhood of $\mathbb{C}^m$ about the origin o for which $f_1 - f_1(o), \dots, f_q - f_q(o)$ are $\mathbb{Q}$-linearly independent. Let r be the rank of the Jacobi matrix $\frac{\partial(f_1,\dots,f_q)}{\partial(z_1,\dots,z_m)}$. Then the transcendence degree of $\mathbb{C}(f_1, \dots, f_q, e^{f_1}, \dots, e^{f_q})$ is greater than or equal to $q + r$.*

Proof. This is [1], Corollary 2. □

Theorem 9. *Let L be a linkage with m helical joints. Let K be an irreducible component of its configuration space containing the initial configuration as a nonsingular point. Let K_0 be the Zariski closure of K in the cylindrical extension of L. Let A be the integral matrix defined by the $\mathbb{Z}$-relations between the helical joint parameters. Then*

$$\dim(K_0) = \dim(K) + m - \operatorname{rank}(A).$$

Consequently, L can be obtained by screw carving from its cylindrical extension.

Proof. By Lemma 7, we have $\dim(K_0) \le \dim(K) + m - \operatorname{rank}(A)$, so it suffices to show the other inequality. Let $d := \dim(K)$, and $q := m - \operatorname{rank}(A)$. Let $\pi_H : K \to \mathbb{C}^m$ be the projection to the helical joint parameters $\alpha_{k_1}, \dots, \alpha_{k_m}$, and let $K_H \subset \mathbb{C}^m$ be its image. Similarly, we define $\pi_C : K_0 \to \mathbb{C}^m \times (\mathbb{P}^1)^m$ as the projection to the cylindrical joint parameters of L' and K_C as the image. There is a natural embedding $K_H \hookrightarrow K_C$, the map π_H is the restriction of π_C along this embedding, and K_C is the Zariski closure of K_H in $\mathbb{C}^m \times (\mathbb{P}^1)^m$.

Let $d_H := \dim(K_H)$. Then there is an analytic isomorphism ϕ of a neighbourhood $U \subset \mathbb{C}^{d_H}$ of the origin o mapping o to the initial configuration. For $k = 1, \dots, m$, let $\overline{\alpha_k} : U \to \mathbb{C}$ be the projection to the joint parameter α_k. They generate a q-dimensional $\mathbb{Q}$-vector space. We may assume that $\overline{\alpha_{k_1}}, \dots, \overline{\alpha_{k_q}}$ generate this vector space. The rank of the Jacobian of $\overline{\alpha_{k_1}}, \dots, \overline{\alpha_{k_q}}$ is equal to the rank of the Jacobian of all coordinate functions, which is equal to d_H. By Theorem 8, the field $\mathbb{C}(\overline{\alpha_{k_1}}, \dots, \overline{\alpha_{k_q}}, e^{\overline{\alpha_{k_1}}}, \dots, e^{\overline{\alpha_{k_q}}})$ has transcendence degree at least $d_H + q$ over $\mathbb{C}$. This field is $\mathbb{C}$-isomorphic to the function field of K_C, by the isomorphism

$$\overline{\alpha_{k_1}} \mapsto g_{k_1}^{-1} s_{k_1}, \dots, \overline{\alpha_{k_q}} \mapsto g_{k_q}^{-1} s_{k_q}, e^{\overline{\alpha_{k_1}}} \mapsto \frac{1 + \mathrm{i} t_{k_1}}{1 - \mathrm{i} t_{k_1}}, \dots, e^{\overline{\alpha_{k_q}}} \mapsto \frac{1 + \mathrm{i} t_{k_q}}{1 - \mathrm{i} t_{k_q}}.$$

Therefore $\dim(K_C) \ge d_H + q$.

Let $E \subset K_C$ be the set of all points x such that $\dim(\pi_C^{-1}(x)) > \dim(K_0) - \dim(K_C)$. Since dimension is upper semicontinuous in the Zariski topology, E is a proper algebraic subvariety of K_C. Since K_H is Zariski dense in K_C, it is not contained in E. Therefore the generic fibre of $\pi_H : K \to K_H$ has dimension $\dim(K_0) - \dim(K_C)$. Hence we have

$$\dim(K_0) = \dim(K_C) + \dim(K) - \dim(K_H) \geq d_H + q + \dim(K) - d_H = \dim(K) + q.$$

□

From the cylindrical extension L_c, we may construct families F_p and F_r of linkages by setting either the rotation parameters or the translation parameters of the joints $j_{k_1}, \ldots, j_{k_q}$ to fixed values $(\tau_{k_1}, \ldots, \tau_{k_q})$ respectively $(\sigma_{k_1}, \ldots, \sigma_{k_q})$, where $\overline{\alpha_{k_1}}, \ldots, \overline{\alpha_{k_q}}$ generate the $\mathbb{Q}$-vector space of all angle parameter functions at the helical joints. This imposes exactly q additional equations to the configuration set K_0, hence the mobility of any linkage in one of the two families is greater than or equal to the mobility of L. In family F_r, every C-joint is replaced by an R-joint, and in family F_p, every C-joint is replaced by a P-joint.

We choose generic members for both families and call them L_p and L_r. The angles and orthogonal distances of neighbouring R-joints is constant for both families, and is equal to the value of the corresponding parameter of L. The offsets do change, but in a transparent way: if j_2 is an H-joint and j_1, j_3 are H- or R-joints in L, then the offset of the axes at the corresponding axes in the family F_r is a linear non-constant function in the family parameter σ_2. Similarly, the angle between the directions of j_1 and j_3 from j_2 in spherical geometry change with the family parameter τ_2. In particular, the generic family member L_r has nonzero offset and the generic family member L_p has nonzero angles at this place.

Linkages with helical joints are called degenerate if there are neighbouring R- or H-joints with equal axes, neighbouring P-joints with equal directions, or an H-joint with a neighbouring P-joint in the direction of the axis of the H-joint. In all these cases it is easily possible to simplify the pair of neighbouring joints from RR to R, HH to H or C, HR to C, PP to P, or HP to C.

Assume that L is nondegenerate. Because the axes/directions of L_r and L_p are equal to the axes/directions of an instance of L_c after application of a motion in K_0, L_r is also nondegenerate, and L_p can only be degenerate if L has neighbouring H-joints with parallel axes. However, it may happen that some joints of L_r or L_p remain fixed during the motion, even if this is not the case for L.

The existence of mobile linkages with only P- and R-joints with particular properties as a consequence of the existence of mobile linkages with H-joints allows to classify the 5-linkages with H-joints. We do that by constructing the linkages L_r and L_p and then compare with the classifications in Section 3 and Theorem 6. We also need the classification of 4-linkages, because it may be that some joints of L_r or L_p are fixed. For convenience, we write here the facts on 4-and 5-linkages that are used below. For the first 4 facts, we assume that L is a nondegenerate mobile linkage with R- and P-joints, such that every joint actually moves. Let n be the number of joints of L.

(1) If $n = 4$ and L has a P-joint, then all axes of R-joints are parallel (this is a special case of Delassus' theorem [6]).
(2) If $n = 4$ and L has no P-joints, then either all axes are parallel, or no neighbouring pair of axes is parallel and all offsets are zero (see [5]).
(3) If $n = 5$ and L has a P-joint, then either all axes of R-joints are parallel, or j_1 is a P-joint, all other joints are of type R, and $h_2 || h_3$ and $h_4 || h_5$ (Theorem 6).
(4) If $n = 5$ and L has no P-joints, then either all axes are parallel, or no neighbouring pair of axes is parallel and at least three of the five offsets are zero (see Section 2).

(5) Any movable CRP linkage is degenerate, i.e., either the axis of the C-joint and the axis of the R-joint coincide, or the axis of the C-joint is parallel to the direction of the P-joint. Here we leave the proof to the reader.

Here is the classification of 5-linkages with joints of type R, P, or C, based on Theorem 9 (compare also with Delassus' classification [6] of 4-linkages with these three types of joints).

Theorem 10. *Let L be a non-degenerate mobile 5-linkage with R-, P-, and H-joints, with at least one H-joint, such that all joints actually move. Up to cyclic permutation, the following cases are possible.*

(1) All axes of R- and H-joints are parallel.
(2) There is one P-joint j_1, all other joints are of type H or R, $h_2||h_3$ and $h_4||h_5$.

Proof. Let r be the number of neighbouring blocks of equal axes of the spherical projection L_s. The proof proceeds by case distinction on r. The cases $r = 4$ and $r = 5$ are split into two subcases.

Case $r = 1$: Then all axes of L_s are equal, hence all axes of H- and R-joints of L are parallel; this is possibility (1) of the theorem.

Case $r = 2$: Then each of the two blocks of R-joints in L_s has at least two joints and at most three joints, because a single joint could not move. The linkage L_r is movable and therefore has at least four joints that actually move. In particular, it cannot happen that all axes of L_r that actually move are parallel. After removing the joints that remain fixed, L_r still has two blocks of parallel axes. By comparing with Facts 3 and 4 above, it follows that L_r is a PRRRR linkage; if, say, j_1 is the prismatic joint, then $h_2||h_3$ and $h_4||h_5$. This is possibility (2) of the theorem.

Case $r = 3$: There is at least one group of joints of the spherical projection L_s with only one R-joint. This joint cannot move. Hence the corresponding H- or R-joint of L does not move either, contradicting our assumption. So this case is impossible.

Case $r = 4$: If L has a P-joint, then L_r is a mobile and nondegenerate PRRRR linkage without any parallel rotation axes. Such a linkage does not exist, hence L has no P-joint. Then we have two parallel neighbouring axis of L. Up to cyclic permutation, we may assume $h_1||h_2$, and the other directions of axes are not parallel. There is no 4R or 5R linkage with exactly two parallel axes (see Fact 3). Therefore at least one of the two joints with parallel axes h_1, h_2 must be fixed in L_r. Without loss of generality, we may assume that L_r is a 4R linkage with axes h_2, h_3, h_4, h_5. By Fact 2, all offsets of L_r are zero. In particular, $o(h_2, h_3, h_4) = o(h_3, h_4, h_5) = 0$. Hence j_3 and j_4 are R-joints in L. Hence h_3 and h_4 are axes of R-joints of L_p. On the other hand, they are not parallel to each other and not parallel to the remaining axes. In any movable 4-linkage or 5-linkage with joints of type R and P, at least one P, any revolute axis is in a block of at least two parallel axes (see Fact 1). It follows that the two joints with axes h_3 and h_4 must be fixed in L_p. Hence L_p has at most three joints that actually move. This is not possible if L_p has an R-joint that actually moves. Since $h_1||h_2$ and h_5 is not parallel to both, the joint of L_p corresponding to j_5 is also fixed, and L_p is a degenerate linkage with two P-joints sharing the same direction. It follows that j_1 and j_2 are H-joints in L. The type of joint j_5 may be either H or R.

Subcase 1: j_5 is an R-joint. The cylindrical extension L_c has two C-joints with axes h_1 and h_2. If the mobility of L_c is 3 or higher, then we freeze one of the two C-joints, and we get mobile CRRR linkage without parallel neighbouring axes, contradicting Fact 1. Therefore the mobility of L_c is 2, and we must have a $\mathbb{Q}$-linear relation between the angle functions. But in L_r, the joint corresponding to j_1 is fixed, and the joint corresponding to j_2 moves. This is a contradiction.

Subcase 2: j_5 is an H-joint. If the mobility of L_c is 2, then we may argue as in Subcase 1: the three angle functions at the C-joints of L_c have to generate a one-dimensional $\mathbb{Q}$-vector space of L_c. Especially, there is a $\mathbb{Q}$-linear relation between the angle functions at the joints corresponding to j_1 and j_2. And in L_r, the joint corresponding to j_1 is fixed, and the joint corresponding to j_2 moves, which gives a contradiction. Hence the mobility of L_c is at least 3. Now we freeze the R-joint with axes h_3 and the translation component at the joint corresponding to j_2. We get a mobile CRRC linkage L' with joint axes h_1, h_2, h_4, h_5. Since $h_1 || h_2$, and h_4 and h_5 have different directions, the spherical projection of L' is fixed at the two joints with axes h_4 and h_5. Hence L' is fixed at the joint with axes h_4 and the C-joint with axes h_5 can be replaced by a P-joint. We may consider L' as a CRP linkage with axes h_1, h_2, and translation direction p_5. By Fact 5, either $h_1 \equiv h_2$ or $h_1 || h_5$. Neither is possible.

Case $r = 5$: Then L_r is either a 5R-linkage or a 4R-linkage with an extra immobile R-joint.

Subcase 1: If L_r is a 5R-linkage, then L_r has at least three vanishing offsets, by Fact 4. Hence L has at least one and at most two H-joints, and L_p has at least one at most two P-joints. Because L_p has no parallel rotation axes, all its R-joints must be fixed. Then L_p must have two P-joints with equal directions. On the other hand, L has no parallel axes. This is a contradiction.

Subcase 2: Without loss of generality, let us say that the joint with axis h_1 is fixed in L_r. By Fact 2, all offsets of L_r are zero. In particular, $o(h_2, h_3, h_4) = o(h_3, h_4, h_5) = 0$. Hence j_3 and j_4 are R-joints. Again, L_p has no parallel rotation axes, so all its R-joints must be fixed. If L_p has two moving P-joints, then they would have to be parallel, which is not possible. Hence L_p has three moving P-joints with direction p_5, p_1, p_2. Then these three vectors have to be linear dependent, which is equivalent to saying that the angle between p_5 and p_2 from p_1 (in spherical geometry) is equal to 0. On the other hand, L_p has a nonzero angle at every joint corresponding to an H-joint of L. This is a contradiction.

□

Example 7. Here is an example that shows that the second case is indeed possible.

Let h_2, h_3 be parallel lines with a distance a to each other. Let h_4, h_5 be another pair of parallel lines, not parallel to the first pair, also with distance a to each other. We assume that $p_2 = -p_3$ and $p_4 = -p_5$. Let $g_2, g_4 \in \mathbb{R}$. For any $\alpha_2 = \alpha_3 \in \mathbb{R}$, the composed motion

$$m_2 m_3 = (1 - \epsilon g_2 \alpha_2 p_2)(1 - \tan\left(\frac{\alpha_2}{2}\right) h_2)(1 - \epsilon g_2 \alpha_3 p_3)(1 - \tan\left(\frac{\alpha_3}{2}\right) h_3)$$

is a translation, where the translation vector lies on a circle in a plane orthogonal to p_2 with radius a. Similarly, for $\alpha_4 = \alpha_5 \in \mathbb{R}$ the composed motion $m_4 m_5$ is a translation with translation vector on another circle with the same radius. We can choose a parametrisation such that the motion $m_2 m_3 m_4 m_5$ is a translation in a fixed direction ($p_2 + p_4$ or $p_2 - p_4$). Hence the linkage with P-joint in this direction and H-joints with axes h_2, h_3, h_4, h_5 and pitches g_2, g_2, g_4, g_4 is mobile.

6. Conclusion

Using Ax's theorem and screw carving, it is possible to investigate mobility questions for arbitrary linkages with helical joints. The classification of mobile closed 5-linkages with joints of type R, P, or H, given in this paper, is just a first application of this reduction.

A challenge for future research is the classification of mobile closed 6-linkages with helical joints. In contrast to the case of 5-linkages, reduction to linkages with joints of type R or P will not be enough, because our knowledge of that linkages is still quite incomplete: even the classification of mobile closed 6R linkages is an open problem. But there is a reason to believe that classifying 6-linkages with at least one H-joint is substantially easier than the 6R case: the linkages constructed by cylindrical extension and fixing either the rotational or the translational parameter have properties that could help the classification (for instance, generic offsets). Another possible attempt would be to extend the theory of bonds to linkages with C-joints and to configuration sets where a fixed set of angle functions satisfy **Z**-linear equations.

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References

[1] J. Ax. On Schanuel's conjectures. *Ann. Math. (2)*, 93:252–268, 1971.
[2] J. E. Baker. The single screw reciprocal to the general plane-symmetric six-screw linkage. *J. Geom. Graph.*, 1:5–12, 1997.
[3] D. J. Bates, C. Peterson, A. J. Sommese, and C. W. Wampler. Numerical computation of the genus of an irreducible curve within an algebraic set. *J. Pure Appl. Algebra*, 215(8):1844–1851, 2011.
[4] G. T. Bennett. The skew isogram mechanism. *Proc. LMS*, 13(2nd Series):151–173, 1913–1914.
[5] K. Brunnthaler, H.-P. Schröcker, and M. Husty. A new method for the synthesis of Bennett mechanisms. In *Proceedings of CK 2005, International Workshop on Computational Kinematics*, Cassino, 2005.
[6] E. Delassus. The closed and deformable linkage chains with four bars. *Bull. Sci. Math.*, 46:283–304, 1922.
[7] S. Di Rocco, D. Eklund, A. J. Sommese, and C. W. Wampler. Algebraic $\mathbb{C}^*$-actions and the inverse kinematics of a general 6R manipulator. *Appl. Math. Comput.*, 216(9):2512–2524, 2010.
[8] P. Dietmaier. *Einfach übergeschlossene Mechanismen mit Drehgelenken.* Habilitation thesis, Graz University of Technology, 1995.
[9] Florian Geiß and Frank-Olaf Schreyer. A family of exceptional Stewart-Gough mechanisms of genus 7. In *Interactions of classical and numerical algebraic geometry*, volume 496 of *Contemp. Math.*, pages 221–234. Amer. Math. Soc., Providence, RI, 2009.
[10] G. Hegedüs, J. Schicho, and H.-P. Schröcker. The theory of bonds: A new method for the analysis of linkages. *Mechanism and Machine Theory*, 70(0):407 – 424, 2013.
[11] M. Husty and H.-P. Schröcker. Algebraic geometry and kinematics. In I. Z. Emiris, F. Sottile, and Th. Theobald, editors, *Nonlinear Computational Geometry*, volume 151 of *IMA Vol. Math. Appl.*, pages 85–107. Springer, 2010.
[12] A. Karger. Classification of 5R closed kinematic chains with self mobility. *Mech. Mach. Th.*, pages 213–222, 1998.
[13] C-C. Lee, C. Stammers, and G. Mulineux. On the historical overview of geometric algebra for kinematics of mechanisms. *International Symposium on History of Machines and Mechanisms*, pages 21 – 34, 2009.
[14] J. Selig. *Geometric Fundamentals of Robotics.* Springer, 2005.
[15] C. W. Wampler and A. J. Sommese. Numerical algebraic geometry and algebraic kinematics. *Acta Numer.*, 20:469–567, 2011.

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