Adaptively Refined Multilevel Spline Spaces from Generating Systems

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Abstract

The truncated basis of adaptively refined multilevel spline spaces was introduced by Giannelli et al. (2012, 2013). It possesses a number of advantages, including the partition of unity property, decreased support of the basis functions, preservation of coefficients and strong stability, that may make it highly useful for geometric modeling and numerical simulation. We generalize this construction to hierarchies of spaces that are spanned by generating systems that potentially possess linear dependencies. This generalization requires a modified framework, since the existing approach relied on the linear independence of the functions generating the spaces in the hierarchy. Many results, such as the preservation of coefficients, can be extended to the more general setting. As applications of the modified framework, we introduce a hierarchy of hierarchical B-splines, which enables us to perform local refinement in the presence of features, and we also extend the adaptive multilevel framework to spaces spanned by Zwart-Powell (ZP) elements, which are special box splines defined on the criss-cross grid. In the latter case we show how to identify the linear dependencies that are present in the truncated hierarchical generating system and use this result to perform adaptive surface fitting with multilevel ZP elements.

Keywords: multilevel spline space, generating system, preservation of coefficients, Zwart-Powell element

1. Introduction

The study of adaptively refined multilevel spline spaces, such as hierarchical B-splines, has a long history in Computer Aided Geometric Design (Forsey and Bartels, 1988; Greiner and Hormann, 1997; Kraft, 1997; Pulli and Lounsbery, 1997). Recently, the investigation of such spaces has gained significant momentum, partially due to their importance for isogeometric analysis (IgA) (see Hughes et al., 2005; Cottrell et al., 2009), where generalizations of tensor-product splines are needed in order to enable localized mesh refinement. Besides the use of T-splines (Dörfel et al., 2010; Li et al., 2012) and of the recently introduced

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Locally Refined (LR) splines of Dokken et al. (2013), approaches that rely on hierarchical techniques have been employed successfully to address the issue of local refinement in IgA (e.g. Vuong et al., 2011; Kuru et al., 2014; Bornemann and Cirak, 2013; Schillinger et al., 2012).

So far, the construction and analysis of adaptively refined multilevel spline spaces has been restricted almost entirely to spaces spanned by *basis functions*, which – by the very definition of this notion – possess the property of linear independence. The aim of the present paper is to provide the generalization to spaces obtained as the span of certain *generating systems* that are potentially linearly dependent.

More precisely, we extend the framework of truncated hierarchical bases (Giannelli et al., 2012, 2013; Speleers et al., 2009) to the case of generating systems. Since the existing approach relies heavily on the property of linear independence, we need to formulate a modified framework (including a different notation) for this generalization. We show that many of the advantageous properties of truncated hierarchical bases can be extended to the new setting. These include the preservation of coefficients, the nested nature of refinement via domain enlargement, and the partition of unity property.

We also consider two applications of the generalized setting. First, we introduce a hierarchy of hierarchical B-splines that enables us to perform local refinement in the presence of features (local modifications). Second, we consider the adaptive multilevel framework to spaces spanned by Zwart-Powell (ZP) elements, which are special box splines defined on the criss-cross grid (see de Boor et al., 1993; Wang, 2001).

This highly interesting class of functions was studied extensively in approximation theory (Lyche et al. (2008), Foucher and Sablonnière (2008); see also Prautzsch and Boehm (2002) for additional information on box splines). ZP elements achieve C^1 smoothness with piecewise quadratic functions and possess a single non-trivial linear dependency relation. We show how to analyze the linear dependencies that are inherited by the generating system that spans the adaptively refined multilevel space. The capabilities of the new construction for adaptively refined multilevel ZP elements are demonstrated by using them for surface fitting, where elimination of these linear dependencies is essential.

The remainder of the paper consists of six parts. First we present the generalized hierarchical construction in Section 2. The third section discusses the property of preservation of coefficients and its consequences. Section 4 derives our results concerning refinement via space and/or domain enlargement. It also presents the first application of the generalized setting to define hierarchies of hierarchical B-splines, which allow to perform adaptive refinement in the presence of features. After briefly discussing the case of generating systems with the property of local linear independence in Section 5, we show how to identify and eliminate the linear dependencies that are inherited by a truncated hierarchical generating system for a certain class of generating systems (which includes hierarchical ZP elements). Based on these results, we present an application to surface fitting using adaptively refined hierarchical ZP elements. Finally, we conclude the paper and identify topics for future research.

2. Adaptive refinement based on generating systems

This section shows how to generalize the construction of truncated hierarchical splines to the case of generating systems, where linear dependencies may or may not be present. We start with some preliminary remarks concerning the notation and establish the selection procedure that leads to the hierarchical construction. Further, we explain the truncation mechanism and introduce the notation of the truncated hierarchical generating system.

2.1. Preliminaries

Throughout this paper we consider spaces V of functions defined on an open set $\Omega \subset \mathbb{R}^D$ that are spanned by finite[1] *generating systems* G. The generating systems are represented by column vectors

$$G = (\gamma_1, \dots, \gamma_n)^T, \tag{1}$$

where the positive integer n denotes the number of functions in G. We will use *row vectors* c to collect the coefficients of functions in

$$V = \operatorname{span} G = \{c\, G \;:\; c \in \mathbb{R}^D\}.$$

We assume that all functions γ_i are non-negative and different from the null function. Generating systems that form a partition of unity,

$$\mathbb{1}\, G = 1,$$

with $\mathbb{1} = (1, \dots, 1)$, will be said to be *normalized*.

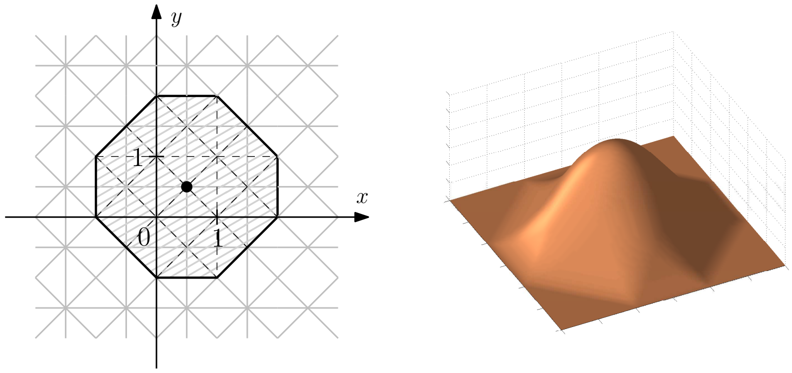
Example 1. As an example of a generating system with a linear dependency relation we consider translates of Zwart-Powell (ZP) element, a C^1 piecewise quadratic box spline. It is defined over a four directional criss-cross grid and it is obtained by applying convolutions to the characteristic function of the unit square in four directions in $\mathbb{R}^2$: $(1,0)$, $(0,1)$, $(1,1)$ and $(-1,1)$. Its support has the shape of an octagon as shown in Figure 1. Hence, the function can be associated with the central point of its support. The translates with respect to the integer grid $\mathbb{Z}^2$ form a generating system with one linear dependency relation. It is known that removing one function restores linear independence. For further information we refer to the monographs of de Boor et al. (1993) and Wang (2001).

We will denote the ZP element in the original position by $\varphi(x)$. The associated point is $(1/2, 1/2)$, i.e. the center of its support. The support and the associated point are depicted in Figure 1(a). The function itself is shown in Figure 1(b).

The translates of φ are

$$\varphi_{\mathbf{i}}(\cdot) := \varphi(\cdot - \mathbf{i}), \quad \mathbf{i} \in \mathbb{Z}^2.$$

[1] Even though we consider finite generating systems, the following theory can also be applied to locally finite systems (see Davydov et al., 1997, for the definition of this notion).



(a) (b)

Figure 1: (a) The support of the Zwart-Powell element φ with associated point shown as black circle and (b) the graph of the function φ.

Given an open set Ω, we consider only those translates that do not vanish on Ω and collect their indices in a set $\mathcal{I}$. These translates then form a finite generating system

$$G = (\varphi_{\mathbf{i}})_{\mathbf{i}\in\mathcal{I}}. \tag{2}$$

Obviously, we can introduce a suitable numbering and obtain a column vector of the form (1). The linear dependency relation of ZP elements will be discussed later in Example 22.

Note that the support of each element of the generating system G is always a subset of Ω, since Ω is the domain of all functions that we consider. Those parts of the octagonal (untrimmed, i.e. not restricted to Ω) supports of the translates that are not contained in Ω will simply be ignored. $\diamondsuit$

We define a relation between generating systems.

Definition 2. Considering two different generating systems G and $\hat{G}$, the generating system G is said to be *coarser* than the generating system $\hat{G}$ (and, conversely, $\hat{G}$ is *finer* than G),

$$G \prec \hat{G},$$

if there exists a non-negative matrix S (i.e. a matrix with non-negative entries) such that

$$G = S\,\hat{G}. \tag{3}$$

Since the generating systems $G, \hat{G}$ and the matrix S are non-negative, the relation implies the following simple observation: Consider the row $(s_{ij})_j$ in the matrix S that contains the coefficients of γ_i with respect to $\hat{G}$,

$$\gamma_i = (c_{ij})_{j=1,\dots,\hat{n}}\;\hat{G}, \tag{4}$$

where $\hat{n}$ is the dimension of $\hat{G}$. Then $c_{ij} \neq 0$ implies $\operatorname{supp}\hat{\gamma}_j \subseteq \operatorname{supp}\gamma_i$, where $\hat{G} = (\hat{\gamma}_1, \dots, \hat{\gamma}_{\hat{n}})^T$. Thus, each function γ_i of G has a representation with respect to $\hat{G}$ that involves only functions with support contained in supp γ_i. Since the systems are not assumed to be linearly independent, several representations with this property may exist.

We summarize several simple properties of the relation $\prec$.

Lemma 3. *1. If $G \prec \hat{G}$ then* span $G \subseteq$ span $\hat{G}$.

2. The relation $\prec$ is reflexive, transitive and weakly antisymmetric in the sense that $G \prec \hat{G}$ and $\hat{G} \prec G$ imply that span $G =$ span $\hat{G}$.

3. If the elements of G form a subset of the elements of $\hat{G}$, then $G \prec \hat{G}$.

To show that $\prec$ is not antisymmetric consider two generating systems $G = (f)$ and $\hat{G} = (\lambda f)$, where $\lambda > 1$ and f is an arbitrary function; these two systems are different but related in both directions, i.e. $G \prec \hat{G}$ and $\hat{G} \prec G$.

2.2. Hierarchical generating systems

Hierarchical generating systems are defined with the help of a *domain hierarchy* and a *hierarchy of generating systems.* The domain hierarchy is simply a decreasing sequence of $N + 1$ subsets $\Omega^\ell \subset \mathbb{R}^D$,

$$\Omega = \Omega^0 \supseteq \Omega^1 \supseteq \cdots \supseteq \Omega^{N+1} = \emptyset, \tag{5}$$

where we added the final domain $\Omega^{N+1} = \emptyset$ in order to simplify the notation later on.

The domains define the complementary hierarchy

$$\Delta^\ell = \Omega \setminus \Omega^{\ell+1} \tag{6}$$

satisfying

$$\Delta^0 \subseteq \Delta^1 \subseteq \cdots \subseteq \Delta^N = \Omega.$$

The elements of this hierarchy will be denoted as *rings*, because, conceptually, Δ^ℓ consist of Ω with the 'hole' defined by $\Omega^{\ell+1}$ (even though it need not be ring-shaped).

The hierarchy of generating systems is a sequence of $N+1$ generating systems $G^\ell = (\gamma_i^\ell)_{i=1,\dots,n^\ell}$ that is monotonically increasing with respect to relation $\prec$,

$$G^0 \prec G^1 \prec \cdots \prec G^N. \tag{7}$$

Denoting their spans with $V^\ell =$ span G^ℓ, we obtain a nested sequence of spaces,

$$V^0 \subseteq V^1 \subseteq \cdots \subseteq V^N.$$

Without loss of generality we will assume that each domain Ω^ℓ is the union of supports of a certain subset of the functions in $G^{\ell+1}$. Considering domain hierarchies without this property does not lead to additional hierarchical generating systems.

In order to prepare the construction of the hierarchical generating systems, we first assume their ordering according to the chosen domain hierarchy. More precisely, such ordering allows for splitting the vectors G^ℓ into three sub-vectors G^ℓ_A, G^ℓ_B, G^ℓ_C with dimensions n^ℓ_A, n^ℓ_B, n^ℓ_C, respectively. The sub-vectors are defined as follows:

$$G^\ell = (\gamma_i^\ell)_{i=1,\ldots,n^\ell} = \begin{pmatrix} G^\ell_A \\ G^\ell_B \\ G^\ell_C \end{pmatrix} = \begin{pmatrix} (\gamma_i^\ell)_{i=1,\ldots,n^\ell_A} \\ (\gamma_i^\ell)_{i=n^\ell_A+1,\ldots,n^\ell_A+n^\ell_B} \\ (\gamma_i^\ell)_{i=n^\ell_A+n^\ell_B+1,\ldots,n^\ell_A+n^\ell_B+n^\ell_C} \end{pmatrix} \tag{8}$$

such that

$$\begin{aligned} i \le n^\ell_A \quad &\Leftrightarrow \quad \operatorname{supp}\gamma_i^\ell \not\subseteq \Omega^\ell, \\ i > n^\ell_A + n^\ell_B \quad &\Leftrightarrow \quad \operatorname{supp}\gamma_i^\ell \subseteq \Omega^{\ell+1}, \text{ and} \\ n^\ell \quad &= \quad n^\ell_A + n^\ell_B + n^\ell_C. \end{aligned} \tag{9}$$

If the support of a function γ_i^ℓ possesses a non-empty intersection with the ring $\Delta^{\ell-1}$, then it belongs to G^ℓ_A. If it possesses a non-empty intersection with Δ^ℓ, but not with $\Delta^{\ell-1}$, then it belongs to G^ℓ_B. All other functions are elements of G^ℓ_C. The partition of G^ℓ is shown in Figure 2. Note that some of these sub-vectors may not be present (void) at certain levels. In particular, G^0_A is void, since the supports of all functions are contained in $\Omega = \Omega^0$, and G^N_C are void since $\Omega^{N+1} = \emptyset$.

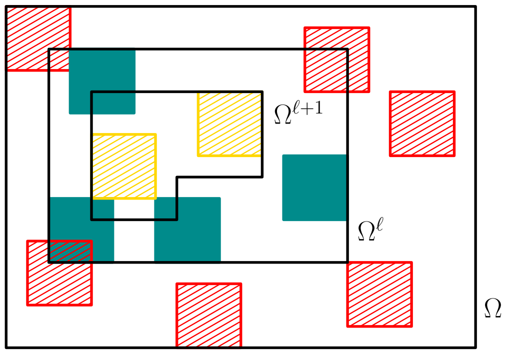


Figure 2: Partition of G^ℓ into G^ℓ_A (red), G^ℓ_B (blue) and G^ℓ_C (yellow).

Since the generating systems form an increasing sequence (7), there exist matrices $S^{\ell+1}$ with non-negative entries, which relate them in terms of Definition 2,

$$G^\ell = S^{\ell+1}\, G^{\ell+1}, \quad \ell = 0, \ldots, N-1. \tag{10}$$

These equations will be called the *refinement relations*. The matrices are denoted by S because of their relation to subdivision methods (e.g. for box splines).

Example 4. Subdivision for box splines is already known and has been well studied (de Boor et al., 1993; Prautzsch and Boehm, 2002). We will consider the refinement relation

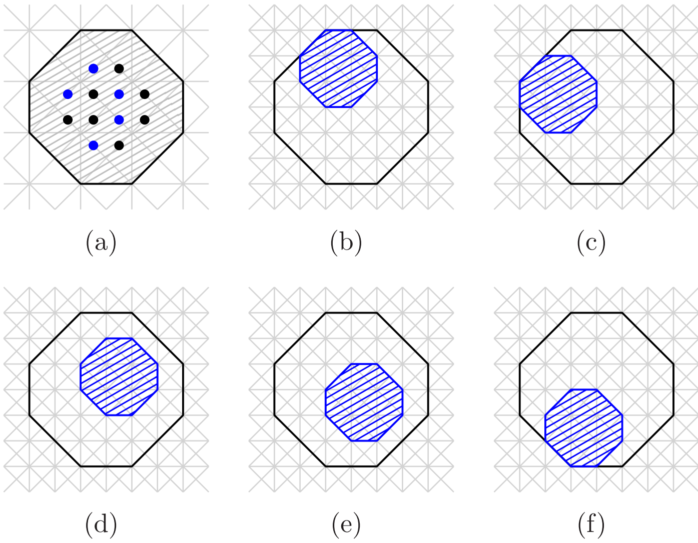


Figure 3: Subdivision of (a) ZP element into twelve finer functions, denoted with black and blue dots; the supports depicted in (b)-(f) correspond to functions denoted with blue dots in (a).

for ZP elements, where a coarse function is a linear combination of twelve functions defined on a finer grid (the supports of some of these finer functions are depicted in Figure 3).

We will consider a nested sequence of domains in $\mathbb{R}^2$ and a nested sequence of generating systems of ZP elements on the grids obtained by refinement steps that split each elementary square into four smaller ones (note that there are two systems of elementary squares on the criss-cross grid). The ZP elements on a grid after ℓ refinement steps are

$$\varphi^\ell_{\mathbf{i}}(x) = \varphi(2^\ell\, x - \mathbf{i}).$$

According to the choice of the domain hierarchy we obtain finite index sets $\mathcal{I}^\ell$, such that the functions $\varphi^\ell_{\mathbf{i}}$ do not vanish on Ω for $\mathbf{i} \in \mathcal{I}^\ell$. We then define the generating systems

$$G^\ell = (\varphi^\ell_{\mathbf{i}})_{\mathbf{i}\in\mathcal{I}^\ell}. \tag{11}$$

The coefficients in the refinement relation between a coarse and twelve finer functions are shown in Figure 4. Note that not all finer functions need to be present at the next level, since some of them might vanish on the domain Ω. $\diamondsuit$

Taking (8) into account, we obtain 9 sub-matrices,

$$G^\ell = \begin{pmatrix} G^\ell_A \\ G^\ell_B \\ G^\ell_C \end{pmatrix} = \begin{bmatrix} S^{\ell+1}_{AA} & S^{\ell+1}_{AB} & S^{\ell+1}_{AC} \\ S^{\ell+1}_{BA} & S^{\ell+1}_{BB} & S^{\ell+1}_{BC} \\ \mathbf{0} & S^{\ell+1}_{CB} & S^{\ell+1}_{CC} \end{bmatrix} \begin{pmatrix} G^{\ell+1}_A \\ G^{\ell+1}_B \\ G^{\ell+1}_C \end{pmatrix} = S^{\ell+1} G^{\ell+1}. \tag{12}$$

Note that $S^{\ell+1}_{CA} = \mathbf{0}$ (the null matrix), since the functions in $G^{\ell+1}_A$ cannot contribute to the values of functions in G^ℓ_C, cf. Eq. (4).

The following result follows directly from the last row in (12):

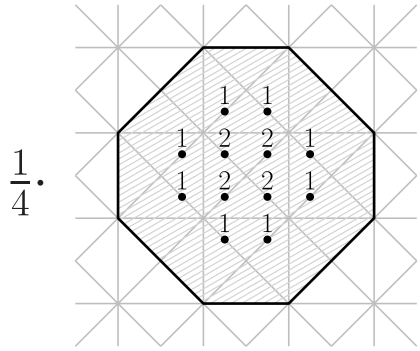


Figure 4: The coefficients corresponding to the twelve finer functions in the refinement relation.

Lemma 5. *We may express G_C^ℓ with respect to all G_B^k, $k \geq \ell + 1$,*

$$G_C^\ell = \sum_{k=\ell+1}^{N-1} \left(\prod_{p=\ell+1}^{k-1} S_{CC}^p \right) S_{CB}^k G_B^k.$$

We are now ready to describe the selection mechanism of the hierarchy, which generalizes the definition in Vuong et al. (2011) to generating systems.

Definition 6. The vector of functions

$$\mathcal{K} = (G_B^\ell)_{\ell=0,\ldots,N}$$

is called the *hierarchical generating system*. Its span is called the *hierarchical space*.

Example 7. We consider the hierarchical generating system with respect to generating systems (11) for $\ell = 0, 1$. The domain hierarchy is depicted in Figure 5. The selected functions from level 0 and level 1 are denoted by circles that are placed at the associated points. Note that some of these points are not contained in Ω, since we use the center point of the untrimmed supports (not restricted to Ω). $\diamondsuit$

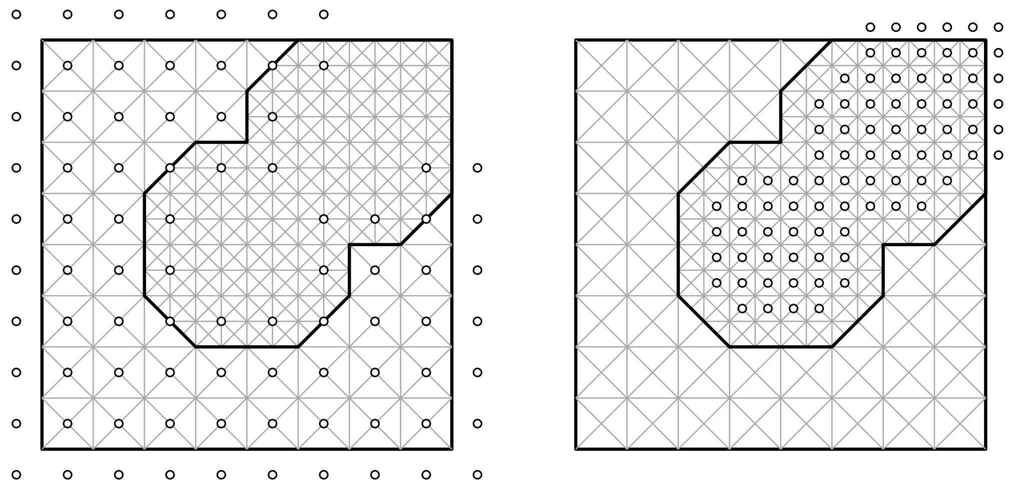
2.3. Truncated hierarchical generating systems

In this section we generalize the construction of truncated hierarchical B-splines of Giannelli et al. (2012, 2013) to the case of generating systems. We consider the same setting as in the previous section.

The *truncation* of a function $f \in V^\ell$ with respect to the next generating system $G^{\ell+1}$ is based on the following simple observation: The values of f on the ring Δ^ℓ are fully determined by its coefficients with respect to $G_A^{\ell+1}$,

$$f = c_A^{\ell+1} G_A^{\ell+1} + c_B^{\ell+1} G_B^{\ell+1} + c_C^{\ell+1} G_C^{\ell+1} \overset{[\text{on } \Delta^\ell]}{=} c_A^{\ell+1} G_A^{\ell+1}.$$

The right-hand side of this equation represents the truncation with respect to generating system $G^{\ell+1}$ and domain $\Omega^{\ell+1}$. It is obtained by removing the contributions of all functions



(a) (b)

Figure 5: Hierarchical generating system of ZP elements with two levels; selected functions from (a) level 0 and (b) level 1.

$\gamma_i^{\ell+1}$ that are selected (i.e. included into the hierarchical generating system $\mathcal{K}$), while leaving the values of f on Δ^ℓ unchanged. More precisely, we omit the contributions of functions from $G_B^{\ell+1}$ that have been selected directly. Moreover, we also omit the contributions of the functions from $G_C^{\ell+1}$ that have been selected indirectly, as they can be represented by functions that are selected at higher levels, see Lemma 5.

Repeatedly applying truncation to the functions G_B^ℓ that were selected for the hierarchical generating system $\mathcal{K}$ gives

$$G_B^\ell \stackrel{[\text{on } \Delta^\ell]}{=} S_{BA}^{\ell+1} G_A^{\ell+1} \stackrel{[\text{on } \Delta^{\ell+1}]}{=} S_{BA}^{\ell+1} S_{AA}^{\ell+2} G_A^{\ell+2} \stackrel{[\text{on } \Delta^{\ell+2}]}{=} \dots \stackrel{[\text{on } \Delta^N]}{=} S_{BA}^{\ell+1} \left(\prod_{k=\ell+2}^{N} S_{AA}^k \right) G_A^N, \quad (13)$$

where we used the refinement equations (12). Finally we obtain the functions

$$T^\ell = S_{BA}^{\ell+1} \left(\prod_{k=\ell+2}^{N} S_{AA}^k \right) G_A^N, \quad \ell = 0, \dots, N-1. \quad (14)$$

Additionally, we define

$$T^N = G_B^N.$$

Definition 8. The vector

$$\mathcal{T} = (T^\ell)_{\ell=0,\dots,N}$$

is called the *truncated hierarchical generating system*.

Any function $\tau \in T^\ell$ is obtained by repeatedly truncating a function $\beta \in G_B^\ell$. We say that β is the *mother* of τ and τ is the *child* of β. Any function $\tau \in T^\ell$ is derived from its corresponding mother $\beta \in G^\ell_B$ by subtracting the contributions of functions in finer generating systems whose children are included in the truncated hierarchical generating system. Consequently, the support of $\beta - \tau$ is contained in $\Omega^{\ell+1}$ and

$$\beta \overset{[\text{on } \Delta^\ell]}{=} \tau.$$

Written in vector notation,

$$G^\ell_B \overset{[\text{on } \Delta^\ell]}{=} T^\ell. \tag{15}$$

Also note that according to (13), the restriction of T^ℓ to Δ^p for $p > \ell$ can be expressed as

$$T^\ell \overset{[\text{on } \Delta^p]}{=} S^{\ell+1}_{BA} \left(\prod_{r=\ell+2}^{p} S^r_{AA} \right) G^p_A. \tag{16}$$

Example 9. According to the refinement relation described in Example 4 we can truncate the ZP element with respect to higher levels. Figure 6 shows truncation of a function in G^ℓ with respect to level $\ell + 1$. The boundary of $\Omega^{\ell+1}$ is depicted by the dashed line. Figure 6(b) shows how the support of a function after truncation becomes smaller. A similar observation was made by Giannelli et al. (2012) for the case of truncated hierarchical B-splines. $\diamondsuit$

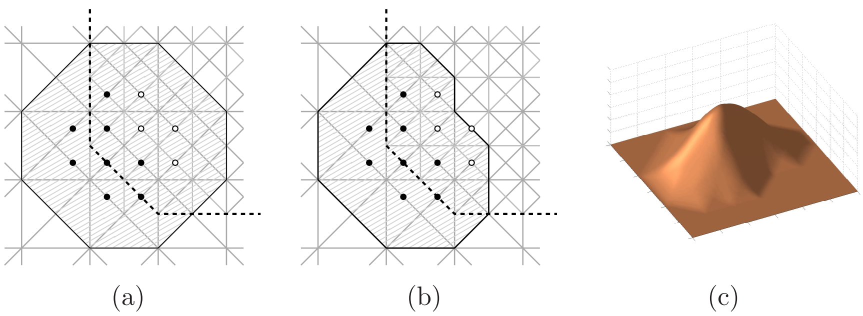


Figure 6: Truncation of a ZP element in G^ℓ with respect to level $\ell+1$; the support (a) before and (b) after truncation. The dashed line is the boundary of $\Omega^{\ell+1}$, circles denote the selected functions from $G^{\ell+1}$ and dots denote the non-selected ones. (c) The truncated ZP element.

3. Preservation of coefficients

The preservation of coefficients is a useful property of truncated hierarchical generating systems. It has already been described by Giannelli et al. (2013) for the case of bases G^ℓ, but its proof relied heavily on their linear independence. Here we generalize it to the case of generating systems. We observe that it requires an additional compatibility condition (18) between the representations of a function at different levels.

Theorem 10 (Preservation of coefficients)**.** *Consider a function f that possesses representations of the form*

$$f \stackrel{[\text{on } \Delta^\ell]}{=} c_A^\ell G_A^\ell + c_B^\ell G_B^\ell, \quad \ell = 0, \ldots, N, \tag{17}$$

where the coefficients of the different levels satisfy the compatibility conditions

$$c_A^{\ell+1} = c_A^\ell S_{AA}^{\ell+1} + c_B^\ell S_{BA}^{\ell+1}, \quad \ell = 0, \ldots, N-1. \tag{18}$$

Then $f \in \operatorname{span} \mathcal{T}$ and

$$f = \sum_{\ell=0}^{N} c_B^\ell T^\ell = (c_B^0, c_B^1, \ldots, c_B^N)\, \mathcal{T} \tag{19}$$

on the entire domain Ω.

The coefficient corresponding to $\tau \in \mathcal{T}$ is the same as the coefficient associated with its mother $\beta \in G_B^\ell$ in the representation (17) of f on Δ^ℓ with respect to G^ℓ. Thus, the truncated hierarchical generating system *preserves the coefficients of mother functions.*

Proof. We start with the representation of f at the finest level,

$$f = c_A^N G_A^N + c_B^N G_B^N,$$

on $\Delta^N = \Omega$, where G_C^N is not present. Repeatedly using (18) we replace the coefficients c_A^ℓ until we arrive at level $\ell = 0$,

$$\begin{aligned} f &= c_A^{N-1} S_{AA}^N G_A^N + c_B^{N-1} S_{BA}^N G_A^N + c_B^N G_B^N \\ &= c_A^{N-2} S_{AA}^{N-1} S_{AA}^N G_A^N + c_B^{N-2} S_{BA}^{N-1} S_{AA}^N G_A^N + c_B^{N-1} S_{BA}^N G_A^N + c_B^N G_B^N = \ldots \\ &= \sum_{\ell=0}^{N-1} c_B^\ell \left(S_{BA}^{\ell+1} \left(\prod_{k=\ell+2}^{N} S_{AA}^k \right) G_A^N \right) + c_B^N G_B^N. \end{aligned}$$

Note that the last step does not produce a term involving c_A^0, since G_A^0 is not present. The theorem then follows directly from the definition of truncated hierarchical generating system. □

Example 11. Due to Theorem 10, the functions in the truncated hierarchical generating system from Example 7 (ZP elements) can be associated with the same (Greville-type) points in the domain. Indeed, we can represent the two coordinate functions, x and y, in all levels by simply choosing the coefficient of each function equal to the coordinates of the associated center point of its support. These coefficients satisfy the compatibility conditions and they are therefore inherited by the truncated hierarchical generating system for any choice of the domain hierarchy. It should be noted that the representations of the coordinate functions are not unique, as ZP elements are not linearly independent. However these Greville-type points are the only ones that give linear precision. ◇

The following result describes the relation between the hierarchical generating system $\mathcal{K}$ and the truncated hierarchical generating system $\mathcal{T}$.

Corollary 12. *The hierarchical generating system $\mathcal{K}$ is coarser than the truncated hierarchical generating system $\mathcal{T}$, i.e. $\mathcal{K} \prec \mathcal{T}$. Both generating systems span the same space,* $\operatorname{span} \mathcal{K} = \operatorname{span} \mathcal{T}$. *The matrix relating them is upper triangular and its diagonal elements are all equal to* 1.

Note that $\mathcal{T}$ does not actually span a finer *space* than $\mathcal{K}$. Still, $\mathcal{T}$ is "finer" than $\mathcal{K}$ in the sense of relation $\prec$.

Proof. Repeatedly using refinement relations (10) allows to derive a similar relation between any two hierarchical generating systems,

$$G^\ell = P^{\ell,k}\, G^k, \quad \ell = 0, \dots, N;\ k = \ell, \dots, N,$$

where

$$P^{\ell,k} = \prod_{i=\ell+1}^{k} S^i \quad \text{and in particular} \quad P^{\ell,\ell} = I.$$

The matrices $P^{\ell,k}$ have all non-negative entries. They also possess a block structure of 3×3 matrices analogous to (12), and the blocks will be denoted in a similar way. (However, no blocks vanish in general.) These equations imply

$$G^\ell_B \overset{[\text{on } \Delta^k]}{=} P^{\ell,k}_{BA}\, G^k_A + P^{\ell,k}_{BB}\, G^k_B, \quad k = \ell, \dots, N.$$

In addition, relation (9) gives

$$G^\ell_B \overset{[\text{on } \Delta^k]}{=} 0, \quad k = 0, \dots, \ell - 1.$$

The coefficients of G^ℓ_B at different levels also satisfy the compatibility condition (18). Consequently, we can use Theorem 10 to derive the relation

$$\begin{pmatrix} G^0_B \\ G^1_B \\ \vdots \\ G^{N-2}_B \\ G^{N-1}_B \\ G^N_B \end{pmatrix} = \begin{bmatrix} I & P^{0,1}_{BB} & P^{0,2}_{BB} & \cdots & P^{0,N-1}_{BB} & P^{0,N}_{BB} \\ \mathbf{0} & I & P^{1,2}_{BB} & \cdots & P^{1,N-1}_{BB} & P^{1,N}_{BB} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & I & P^{N-2,N-1}_{BB} & P^{N-2,N}_{BB} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & I & P^{N-1,N}_{BB} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & I \end{bmatrix} \begin{pmatrix} T^0 \\ T^1 \\ \vdots \\ T^{N-2} \\ T^{N-1} \\ T^N \end{pmatrix}$$

between $\mathcal{K}$ and $\mathcal{T}$, where $\mathbf{0}$ and I represent null-matrices and identity matrices of the appropriate dimensions. The matrix appearing in this relation is non-negative, thus $\mathcal{K} \prec \mathcal{T}$. Moreover, the matrix is invertible, as it is upper triangular and the diagonal entries are all 1, therefore $\operatorname{span} \mathcal{K} = \operatorname{span} \mathcal{T}$. □

As another consequence of the preservation of coefficients, we are able to show that the functions in $\mathcal{T}$ sum up to one:

Corollary 13. *For normalized generating systems G^ℓ the truncated hierarchical generating system $\mathcal{T}$ forms a convex partition of unity, provided that the refinement matrices satisfy the compatibility conditions*

$$\mathbb{1}^{\ell+1} = \mathbb{1}^{\ell}\, S^{\ell+1}, \quad \ell = 0, \dots, N-1.$$

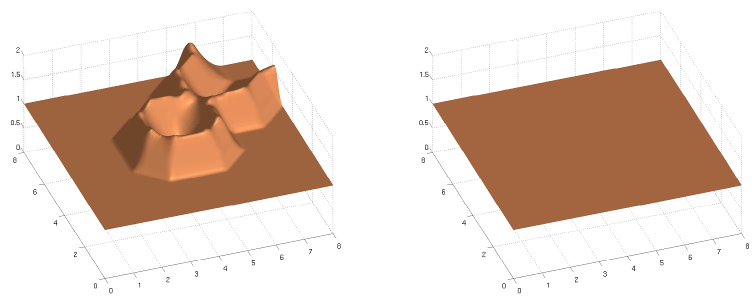
$\mathbb{1}^k$ denotes the row vector of dimension n^k with all elements equal to 1.

Proof. This can be derived using Theorem 10 and considering the representation of the function

$$1 = \mathbb{1}^\ell G^\ell \overset{[\text{on } \Delta^\ell]}{=} \mathbb{1}_A^\ell G_A^\ell + \mathbb{1}_B^\ell G_B^\ell$$

at different levels, where $\mathbb{1}_A^\ell$ and $\mathbb{1}_B^\ell$ are row vectors consisting entirely of ones and of dimensions n_A^ℓ and n_B^ℓ, respectively. Moreover, the functions in $\mathcal{T}$ are non-negative, since they are linear combinations of non-negative functions with non-negative coefficients, cf. (14). □

Example 14. Figure 7 shows the sum of functions in the hierarchical generating system and the truncated hierarchical generating system constructed with respect to the domain hierarchy in Example 7. Observe that the sum for the hierarchical generating system is not equal to 1, since the functions do not form a partition of unity. ♢



(a) (b)

Figure 7: Sums of functions in (a) the hierarchical generating system and in (b) the truncated hierarchical generating system from Example 7.

4. Refinement

We consider two strategies for refining the hierarchical space.

- The first approach consists of enlarging each domain Ω^ℓ in the domain hierarchy to $\hat{\Omega}^\ell \supseteq \Omega^\ell$ (except for $\hat{\Omega}^0 = \Omega^0 = \Omega$ that remains unchanged).

- The second possibility is to replace each generating system G^ℓ by a new system $\hat{G}^\ell$ which is finer with respect to the relation $\prec$. E.g., in the case of tensor-product B-splines, this can be achieved by knot insertion or degree elevation.

Both methods lead to finer hierarchical spaces. First, we consider the enlargement of the domain hierarchy.

Proposition 15. *Consider two nested domain hierarchies*

$$\left.\begin{array}{l}\Omega = \Omega^0 \supseteq \Omega^1 \supseteq \cdots \supseteq \Omega^{N+1} = \emptyset \\ \Omega = \hat{\Omega}^0 \supseteq \hat{\Omega}^1 \supseteq \cdots \supseteq \hat{\Omega}^{N+1} = \emptyset\end{array}\right\} \quad \textit{with} \quad \Omega^\ell \subseteq \hat{\Omega}^\ell, \quad \ell = 1, \ldots, N \tag{20}$$

and the hierarchical generating systems defined on them. The second hierarchical generating system is finer than the first one, $\mathcal{K} \prec \hat{\mathcal{K}}$.

Proof. Enlarging the domain hierarchy changes the definitions of sub-vectors G^ℓ_A, G^ℓ_B and G^ℓ_C of G^ℓ. We denote the sub-vectors with respect to the enlarged domain hierarchy by $\hat{G}^\ell_A$, $\hat{G}^\ell_B$ and $\hat{G}^\ell_C$. Some of the functions from G^ℓ_A may belong to $\hat{G}^\ell_B$ or even to $\hat{G}^\ell_C$, and some of the functions in G^ℓ_B may belong to $\hat{G}^\ell_C$. The relation between the two sets of sub-vectors can be expressed by a permutation matrix Q^ℓ with 9 sub-matrices Q^ℓ_{AA}, Q^ℓ_{AB} etc.,

$$G^\ell = \begin{pmatrix} G^\ell_A \\ G^\ell_B \\ G^\ell_C \end{pmatrix} = Q^\ell \hat{G}^\ell = \begin{bmatrix} Q^\ell_{AA} & Q^\ell_{AB} & Q^\ell_{AC} \\ \mathbf{0} & Q^\ell_{BB} & Q^\ell_{BC} \\ \mathbf{0} & \mathbf{0} & Q^\ell_{CC} \end{bmatrix} \begin{pmatrix} \hat{G}^\ell_A \\ \hat{G}^\ell_B \\ \hat{G}^\ell_C \end{pmatrix}. \tag{21}$$

Note that Q^ℓ_{BA}, Q^ℓ_{CA} and Q^ℓ_{CB} are zero matrices, since functions in G^ℓ_B cannot be expressed with respect to $\hat{G}^\ell_A$ and functions in G^ℓ_C cannot be expressed with respect to $\hat{G}^\ell_B$ and $\hat{G}^\ell_C$.

We now express each sub-vector G^ℓ_B as a non-negative linear combination of the sub-vectors $\hat{G}^k_B$, $k = 0, \ldots, N$. By (21) G^ℓ_B can be written as

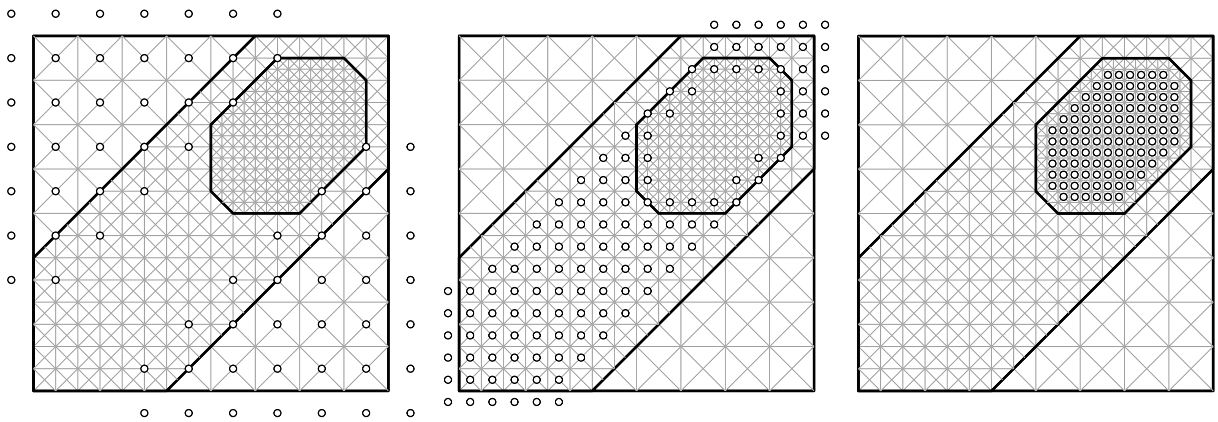
$$G^\ell_B = Q^\ell_{BB} \hat{G}^\ell_B + Q^\ell_{BC} \hat{G}^\ell_C. \tag{22}$$

Using Lemma 5 we replace $\hat{G}^\ell_C$ in the above equation by

$$\hat{G}^\ell_C = \sum_{k=\ell+1}^{N} \Bigg(\prod_{p=\ell+1}^{k-1} \hat{S}^p_{CC} \Bigg) \hat{S}^k_{CB} \hat{G}^k_B,$$

where $\hat{S}^k$ denotes the matrix relating the enlarged generating systems $\hat{G}^k$ and $\hat{G}^{k+1}$. Thus we can write G^ℓ_B as a linear combination of $\hat{G}^k_B$ with non-negative coefficients for $k = \ell, \ldots, N-1$. The matrix connecting $\mathcal{K}$ and $\hat{\mathcal{K}}$ has therefore non-negative entries. ☐

Example 16. Consider the domain hierarchy in Example 7, $\Omega \supseteq \Omega^1 \supseteq \Omega^2 = \emptyset$. By enlarging the domains $\Omega^1 \subset \hat{\Omega}^1$ and $\Omega^2 \subset \hat{\Omega}^2$ we obtain a finer hierarchical generating system. Figure 8 shows the new domain hierarchy and the selected functions from level 0, 1 and 2. They are represented by circles. Notice that adding a domain is just a special case of domain enlargement. ♢



(a) (b) (c)

Figure 8: Domain hierarchy for hierarchical generating system with enlarged domains $\hat{\Omega}^1$ and $\hat{\Omega}^2$ according to Example 7. The selected functions from level 0, 1 and 2 (from left to right) are represented by circles.

Second, we consider the possibility of enlarging the generating systems in terms of Definition 2. The proof of the Proposition 15 can be extended to cover this situation as well. Indeed, the refinement relations between G^ℓ and $\hat{G}^\ell$ can be written in the same form as in (21). The matrices Q^ℓ in this case, however, are no longer just permutation matrices.

Proposition 17. *Consider two sequences of generating systems*

$$\left.\begin{array}{l} G^0 \prec G^1 \prec \cdots \prec G^N \\ \hat{G}^0 \prec \hat{G}^1 \prec \cdots \prec \hat{G}^N \end{array}\right\} \quad \text{with} \quad G^\ell \prec \hat{G}^\ell, \quad \ell = 0, \ldots, N, \tag{23}$$

and the hierarchical generating systems defined by them. The second hierarchical generating system is finer than the first one, $\mathcal{K} \prec \hat{\mathcal{K}}$.

Note that both propositions also imply that

$$\operatorname{span} \mathcal{T} = \operatorname{span} \mathcal{K} \subseteq \operatorname{span} \hat{\mathcal{K}} = \operatorname{span} \hat{\mathcal{T}}. \tag{24}$$

Example 18. We consider the nested spaces spanned by the generating systems G^ℓ that consist of ZP elements from Example 4. In addition, for each generating system G^ℓ, we consider the Bernstein-Bézier basis $\hat{G}^\ell$ of C^0 quadratic splines on the triangulation defined by the ZP elements of that level. It consists of one function for each vertex and one (additional) function for each edge of the triangulation (this corresponds to the six quadratic Bernstein polynomials for each triangle.) When replacing G^ℓ with $\hat{G}^\ell$ we enlarge the hierarchical space defined by the (truncated) hierarchical generating system $\mathcal{K}$ and $\mathcal{T}$. ♢

The combination of both refinement strategies leads to the following General Enlargement Theorem (GET):

Theorem 19 (General Enlargement Theorem – GET)**.** *Consider two sequences of hierarchical domains* (20)*, two sequences of generating systems* (23) *and the hierarchical generating systems defined by them. Then the second hierarchical generating system is finer than the first one.*

The proof follows directly from the previous two propositions. Several remarks on this result are in order.

- As a special case, GET covers the case of adding another level to the hierarchy. In particular, one may consider two hierarchies with

$$\emptyset = \Omega^N \subseteq \hat{\Omega}^N \neq \emptyset.$$

  This also proves that the generating systems which are generated by an inductive definition (as used, e.g. in Giannelli et al. (2012)) form an increasingly finer sequence.

- GET makes it possible to define a *hierarchy of hierarchies* as follows. Consider a sequence of (truncated) hierarchical generating systems $G^k = \mathcal{K}^k$ (or $G^k = \mathcal{T}^k$), $k = 0, \ldots, M$, such that each of them is generated by a hierarchy of domains $\omega^{k,\ell}$ and a hierarchy of generating systems $g^{k,\ell}$, $\ell = 0, \ldots, N$. We assume that these hierarchies are all nested,

$$\omega^{k,\ell} \subseteq \omega^{k,\ell+1}, \quad g^{k,\ell} \prec g^{k,\ell+1}, \quad \ell = 0, \ldots, N-1. \tag{25}$$

  Now, if we consider another global domain hierarchy (5), then we can define a *hierarchy of hierarchical spaces*. This allows for adaptive refinement in the presence of local features, e.g. surfaces with sharp edges and local C^0 continuity[2] (see Example 20), but also for local degree elevation (combinations of adaptive p- and h-refinement).

- GET can be extended to truncated hierarchical generating systems, but this requires an additional condition concerning the compatibility of the refinement relations,

$$S^{\ell+1} Q^{\ell+1} = Q^{\ell} \hat{S}^{\ell+1} \quad \text{for } \ell = 0, \ldots, N-1. \tag{26}$$

  Due to space limitations we do not discuss further details. Note that the spaces are nested even without this assumption.

Example 20. We consider a hierarchical generating system, constructed with respect to nested sequence of two hierarchical B-spline bases $\mathcal{K}^0 \prec \mathcal{K}^1$. Their hierarchical meshes and selected functions are depicted in Figure 9(a) for $\mathcal{K}^0$ and in Figure 9(b) for $\mathcal{K}^1$. We define a surface as a linear combination of basis functions in $\mathcal{K}^0$ with coefficients in $\mathbb{R}^3$. We build a hierarchy of hierarchical spaces to allow for adaptive refinement in a selected region, see Figure 9.

[2]Other techniques for modeling surfaces with sharp features include using T-splines or special subdivision surfaces (Sederberg et al., 2003; Kosinka et al., 2014).

The hierarchical B-spline basis $\mathcal{K}^0$ is obtained from quadratic B-spline bases $g^{0,0} \prec g^{0,1}$ and domains $\omega^{0,0} \supseteq \omega^{0,1}$. Note that $g^{0,1}$ has a double knot in the middle, which results in a surface with a local sharp feature. More precisely, the smoothness of the surface in this region decreases to C^0.

In order to perform adaptive refinement of this particular surface, we find a finer hierarchical generating system by performing space enlargement while leaving the domains unchanged. In particular, we obtain $g^{0,0} \prec g^{1,0}$ and $g^{0,1} \prec g^{1,1}$ by knot insertion. By Theorem 19 we get a nested sequence of hierarchical generating systems $\mathcal{K}^0 \prec \mathcal{K}^1$.

Moreover, we choose a domain hierarchy $\Omega \supseteq \Omega^1$, where $\Omega = \omega^{0,0} = \omega^{1,0}$. The domain Ω^1 is shown in Figure 9 in blue. We construct a hierarchical generating system $\mathcal{K}$ with respect to the domain hierarchy $\Omega \supseteq \Omega^1$ and nested sequence $\mathcal{K}^0 \prec \mathcal{K}^1$. In this special configuration $\mathcal{K}^0$ and $\mathcal{K}^1$ are even locally linearly independent. Thus, $\mathcal{K}$ is linearly independent.

Figures 9(c) and 9(d) show the selected functions from $\mathcal{K}^0$ and $\mathcal{K}^1$ with respect to the domains Ω and Ω^1. Figures 9(e) and 9(f) show the surface before and after local modification. Note that the smoothness of the surface stays the same after modification and the local feature does not extend globally. ♢

5. Linear independence

Clearly, linear independence is a highly desirable property of a generating system, since it is essential for various applications. This section describes how to guarantee it with the help of local linear independence. This approach goes back to the work of Kraft (1997).

Recall that a generating system G is said to be *locally linearly independent*, if for any open subset $U \subseteq \Omega$ all the functions γ_i from G that do not vanish on U are linearly independent on U,

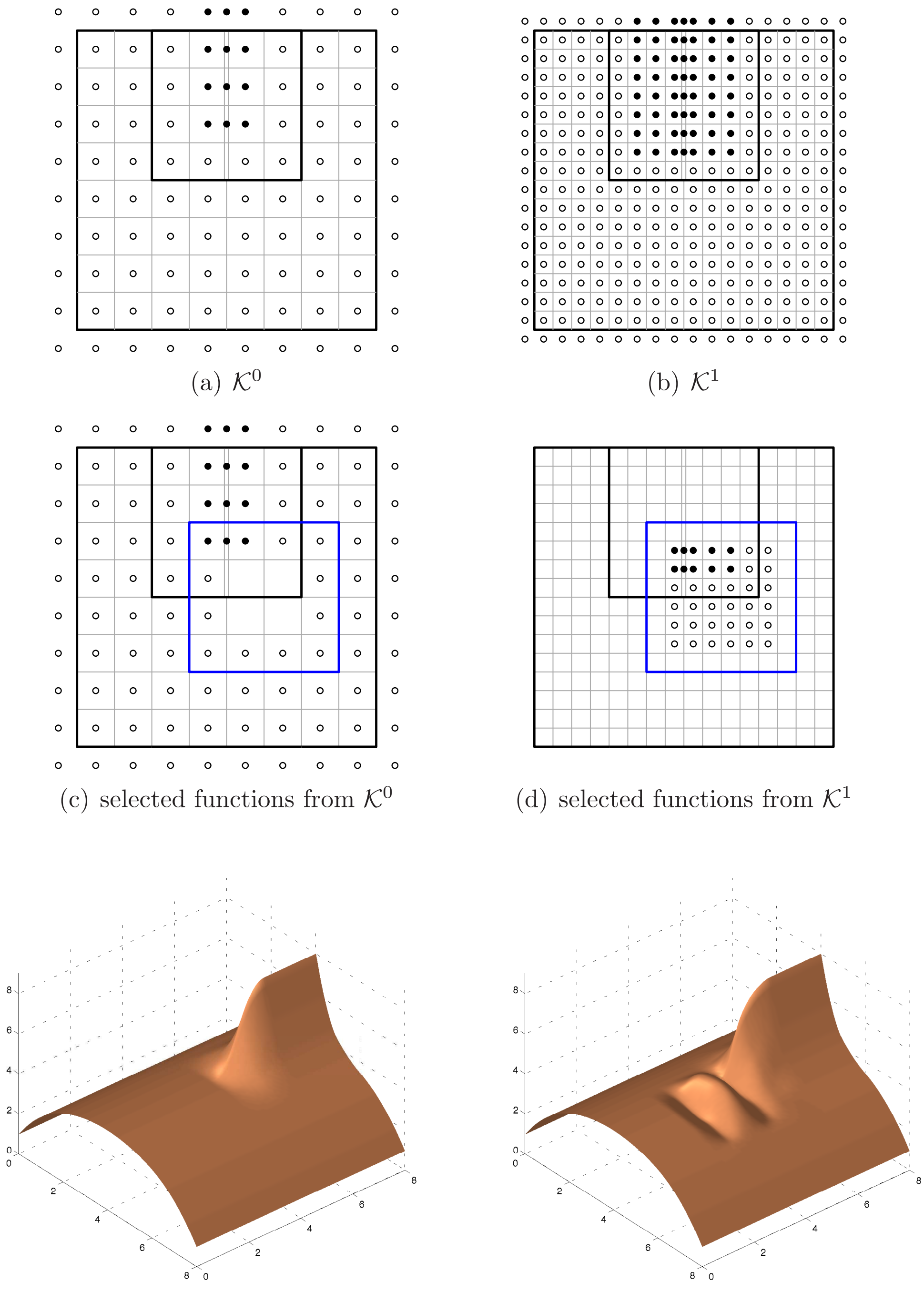
$$\forall j = 0, \ldots, n: \left(0|_U = \sum_{i=0}^{n} c_i \gamma_i|_U \ \wedge \ \gamma_j|_U \not\equiv 0|_U \ \Rightarrow \ c_j = 0 \right). \tag{27}$$

This property allows to conclude linear independence of the hierarchical generating systems and truncated hierarchical generating systems that are derived from domain hierarchies and hierarchies of generating systems.

Proposition 21. *If the generating systems G^ℓ are locally linearly independent, then both the hierarchical generating system $\mathcal{K}$ and the truncated hierarchical generating system $\mathcal{T}$ are linearly independent.*

The first proof of this result for hierarchical generating systems is due to Kraft (1997). For the proof in the case for truncated hierarchical generating systems see Giannelli et al. (2012).

Local linear independence is a stronger requirement than linear independence. For instance, while it is known that B-splines are locally linearly independent, the general



(a) $\mathcal{K}^0$ (b) $\mathcal{K}^1$

(c) selected functions from $\mathcal{K}^0$ (d) selected functions from $\mathcal{K}^1$

(e) surface from span $\mathcal{K}^0$ (f) surface from span $\mathcal{K}$

Figure 9: Hierarchy of hierarchical spaces defined with respect to hierarchical generating systems $\mathcal{K}^0 \prec \mathcal{K}^1$ and hierarchical domains $\Omega \supseteq \Omega^1$. The hierarchical meshes for $\mathcal{K}^0$ and $\mathcal{K}^1$ are shown in (a) and (b), respectively. The boundary of domain Ω^1 is denoted with a blue line in (c) and (d). The surface is shown (e) before and (f) after a local modification.

construction of the hierarchical B-spline basis introduces local dependencies. Even the truncation does not guarantee the absence of local linear dependencies. However, we can maintain the local linear independence of hierarchical B-splines, provided that the underlying spaces and hierarchical domains satisfy further assumptions. For instance, the hierarchical B-splines in Example 20 are locally linearly independent. We plan to perform a more detailed investigation of this issue in the future.

6. Controlling the linear dependencies

The results of the previous section do not apply to generating systems with linear dependencies. For ZP elements it is known that omitting one function from the generating system restores linear independence. In this section we will show how to extend this observation to the hierarchical setting. More precisely, we formulate assumptions that will allow us to characterize the linear dependencies that are present in the truncated hierarchical generating system. These assumptions are satisfied by ZP elements.

Each generating system G defines a linear mapping from $\mathbb{R}^n$ to span G,

$$c \mapsto c\,G, \quad c \in \mathbb{R}^n.$$

The kernel of this mapping is the *space of all linear dependency relations* in G. We denote this space by $\mathcal{L}(G)$. We show how to find a basis for $\mathcal{L}(\mathcal{T})$.

We consider a hierarchy of generating systems that possess exactly one linear dependency relation each, i.e. $\dim \mathcal{L}(G^\ell) = 1$. For each generating system G^ℓ there exists a non-trivial row vector $z^\ell = (z^\ell_1, \ldots, z^\ell_{n^\ell})$, such that $z^\ell G^\ell = 0$. Moreover, we assume that the linear dependency relation involves all functions, i.e. that *all elements of this vector are non-zero.*

Applying the refinement relation (10) to the linear dependency relation of G^ℓ produces a representation of the function $f = 0$ with respect to $G^{\ell+1}$. We make the following simplifying assumption:

(A1) Applying the refinement relation (10) to the linear dependency relation of G^ℓ gives the trivial representation of $f = 0$, i.e.

$$z^\ell S^{\ell+1} = \mathbb{O}^{\ell+1},$$

where $S^{\ell+1}$ is the matrix relating G^ℓ and $G^{\ell+1}$ in terms of relation $\prec$ and $\mathbb{O}^{\ell+1}$ is a row null vector of dimension $n^{\ell+1}$.

Example 22. The generating system of ZP elements possesses one linear dependency relation. The null function can be represented as a linear combination with coefficients in a “chessboard pattern” of alternating coefficients +1 and −1 as shown in Figure 10. Note that this linear dependency relation involves all functions in the generating system. Moreover, this relation satisfies assumption (A1). $\diamondsuit$

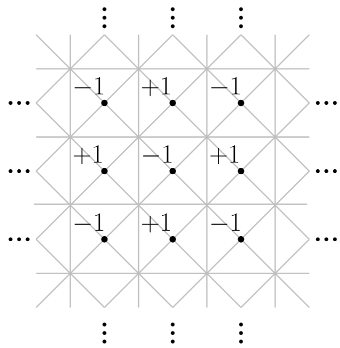


Figure 10: Linear dependency relation in a generating system of ZP elements (the "chessboard pattern"). The picture shows the corresponding coefficients. The functions are represented by dots at the associated points.

Recall that the definition of the truncated hierarchical generating system is based on the hierarchy of domains (5). These define the rings (6). For each generating system G^ℓ we define a connectivity graph C^ℓ as follows:

- All elements of G^ℓ_A and G^ℓ_B are vertices of C^ℓ.
- Two different vertices γ^ℓ_i and γ^ℓ_j are connected by an edge whenever

$$\operatorname{supp} \gamma^\ell_i \cap \operatorname{supp} \gamma^\ell_j \cap \Delta^\ell \neq \emptyset.$$

We denote the vertex sets of the connected components of the connectivity graph by C^ℓ_k, $k = 1, \ldots, M^\ell$, where M^ℓ denotes the number of connected components of level ℓ.

We define G^ℓ-*connected components* of the ring Δ^ℓ,

$$\Delta^\ell_k = \bigcup_{\substack{j=0,\ldots,n^\ell \\ \gamma^\ell_j \in C^\ell_k}} \operatorname{supp} \gamma^\ell_j \cap \Delta^\ell. \tag{28}$$

Note that each Δ^ℓ_k is not necessarily a connected set in $\mathbb{R}^D$. If $j > n^\ell_A$ for all j such that $\gamma^\ell_j \in C^\ell_k$, then both Δ^ℓ_k and C^ℓ_k will be called *orphans of level* ℓ. Thus, *all functions from* G^ℓ_A *vanish (are equal to 0) on an orphan* Δ^ℓ_k.

We assume a special ordering of the vectors G^ℓ as follows: We order the functions in the sub-vector G^ℓ_A with respect to connected components C^ℓ_k for $k = 1, \ldots, M^\ell$. Similarly we order the sub-vector G^ℓ_B. More precisely, after the ordering we can write G^ℓ_A and G^ℓ_B as

$$G^\ell_A = (G^\ell_{A,k})_{k=1,\ldots,M^\ell}, \quad G^\ell_B = (G^\ell_{B,k})_{k=1,\ldots,M^\ell}.$$

Note that some sub-vectors may be void (not present). For instance, $G^\ell_{A,k}$ is not present whenever C^ℓ_k is an orphan. When considering a linear combination $c^\ell\, G^\ell$, where c^ℓ is the row vector of coefficients, the sub-vectors of coefficients corresponding to $G^\ell_{A,k}$ and $G^\ell_{B,k}$ will be denoted by $c^\ell_{A,k}$ and $c^\ell_{B,k}$, respectively. In particular, this also applies to the coefficients of the global linear dependency relations with coefficients z^ℓ.

Example 23. We consider the hierarchical B-spline basis, defined with respect to the univariate cubic B-splines over uniform knot vectors. The hierarchy of bases is obtained by dyadic knot insertion and the domain hierarchy is shown in Figure 11(a). Although connectivity graphs in case of B-splines are irrelevant, since the hierarchical generating system is linearly independent, the example gives the idea how the connectivity graphs are generated.

Figure 11(d) shows the connected components of the connectivity graphs. The functions in G^ℓ_B are represented by shaded circles.

First, we consider the level 0. Since G^0_A is empty, the connectivity graph at level 0 contains only functions from G^0_B. Therefore, every connected component is an orphan. In this particular example, the connectivity graph consists of two components C^0_1 and C^0_2. We obtain two G^0-connected components Δ^0_1 and Δ^0_2, see Figure 11(b).

Second, we consider level 1. The connectivity graph consists of three connected components. Note that C^1_2 contains only functions from G^1_B and is therefore an orphan. We obtain three G^1-connected components of Δ^1 and one of them is an orphan. Note that Δ^1_1 is not a connected set in $\mathbb{R}$.

The connectivity graph at level 2 (not shown) has only one connected component, since $\Delta^2 = \Omega$. It is not an orphan since all functions in G^2_A are present. $\diamondsuit$

In addition to the assumption (A1) about the linear dependency relations, we will also need an assumption concerning the rings Δ^ℓ.

(A2) Each connected component Δ^ℓ_k of the ring Δ^ℓ admits only linear dependency relations that are instances of the global linear dependency relation in G^ℓ. More precisely, if

$$c^\ell_{A,k}\, G^\ell_{A,k} + c^\ell_{B,k}\, G^\ell_{B,k} \overset{[\text{on } \Delta^\ell_k]}{=} 0 \tag{29}$$

then there exists a scalar ϱ, such that

$$c^\ell_{A,k} = \varrho\, z^\ell_{A,k}, \quad c^\ell_{B,k} = \varrho\, z^\ell_{B,k},$$

where $z^\ell_{A,k}$ and $z^\ell_{B,k}$ are sub-vectors of the vector z^ℓ, corresponding to $G^\ell_{A,k}$ and $G^\ell_{B,k}$.

This assumption, which is satisfied by ZP elements, generalizes the notion of local linear independence to generating systems with linear dependencies and will be referred to as *generalized local linear independence.*

A linear dependency relation in $\mathcal{T}$ will be represented by a row vector w. Its sub-vector with respect to T^ℓ will be denoted by u^ℓ,

$$0 = w\,\mathcal{T} = (u^0, \ldots, u^N)\,\mathcal{T} = \sum_{\ell=0}^{N} u^\ell\, T^\ell.$$

Moreover, each u^ℓ consists of sub-vectors u^ℓ_k, one for each G^ℓ-connected component of the ring Δ^ℓ.

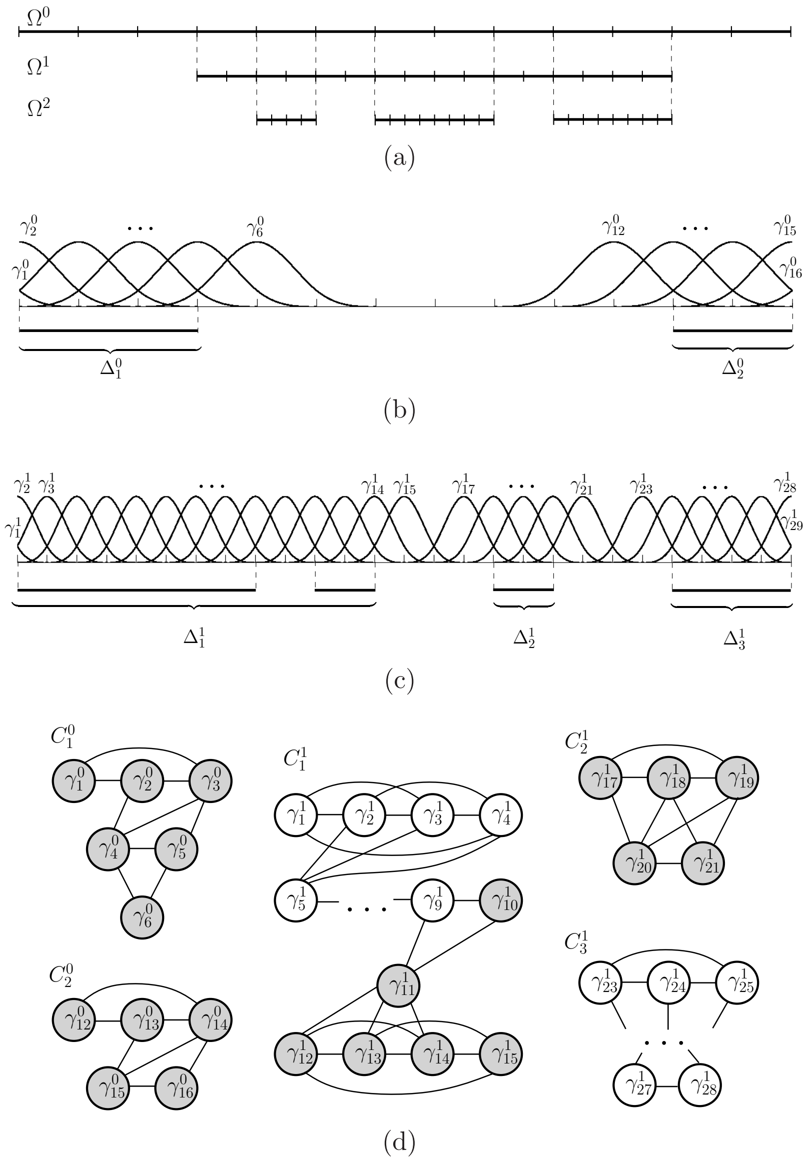


Figure 11: An example of connectivity graphs for univariate cubic hierarchical B-splines with the domain hierarchy, shown in (a). The graphs of levels 0 and 1 are shown in (d); the functions in G^0_B and G^1_B are represented by shaded circles. The corresponding G^0- and G^1-connected components are shown in (b) and (c).

Due to the ordering of G^ℓ_B and truncation process (13), T^ℓ consists of M^ℓ sub-vectors T^ℓ_k such that

$$T^\ell \overset{[\text{on } \Delta^\ell_k]}{=} T^\ell_k.$$

Note that some of the vectors T^ℓ_k may be void.

Let K and L be indices, such that C^L_K is an orphan. We define a linear dependency relation associated with it,

$$w^L_K = (u^0_1, \dots, u^0_{M^0}, \dots, u^N_1, \dots, u^N_{M^N}),$$

where

$$u^\ell_k = \begin{cases} z^\ell_{B,k} & \text{if } \ell = L \text{ and } k = K, \\ 0 & \text{otherwise,} \end{cases} \tag{30}$$

for $\ell = 0, \dots, N$ and $k = 1, \dots, M^\ell$.

Lemma 24. *The vector w^L_K represents a linear dependency relation of $\mathcal{T}$.*

Proof. We consider the representation of $f = 0$ on the ring Δ^ℓ with respect to G^ℓ defined by

$$0 \overset{[\text{on } \Delta^\ell_k]}{=} \mathbb{O}^\ell_{A,k}\, G^\ell_{A,k} + \mathbb{O}^\ell_{B,k}\, G^\ell_{B,k} \tag{31}$$

for each connected component C^ℓ_k, $k = 1, \dots, M^\ell$, $k \neq K$, $\ell = 0, \dots, N$, $\ell \neq L$, and

$$0 \overset{[\text{on } \Delta^L_K]}{=} z^L_{B,K}\, G^L_{B,K}$$

for the orphan C^L_K. Note that representing 0 on the orphan Δ^L_K involves only functions from G^L_B. Using assumption (A1) we can verify that the compatibility condition (18) is satisfied. The lemma now follows from Theorem 10. □

Theorem 25. *The set of linear dependencies*

$$\{w^L_K \,:\, L = 0, \dots, N,\ K = 0, \dots, M^\ell, C^L_K \text{ is an orphan}\} \tag{32}$$

is a basis for the space $\mathcal{L}(\mathcal{T})$ of linear dependency relations in $\mathcal{T}$.

Proof. According to the previous lemma, each w^L_K is a linear dependency relation in $\mathcal{T}$. Due to G^L-connectivity, the non-zero entries of w^L_K have mutually different indices. Therefore, the vectors w^L_K are linearly independent.

We show that these linear dependencies span the entire linear space $\mathcal{L}(\mathcal{T})$. Suppose $w = (v^0, \dots, v^N)$ is a linear dependency relation in $\mathcal{T}$,

$$0 = w\,\mathcal{T} = (v^0, \dots, v^N)\,\mathcal{T} = \sum_{\ell=0}^{N} v^\ell\, T^\ell. \tag{33}$$

We show that w can be expressed as a linear combination of w^L_K. We use induction with respect to level ℓ to express w on each ring Δ^ℓ. Starting with the lowest level $\ell = 0$ we restrict equation (33) to each connected component Δ^0_k. Using equation (15) we obtain

$$0 \overset{[\text{on } \Delta^0_k]}{=} w\,\mathcal{T} \overset{[\text{on } \Delta^0_k]}{=} v^0_k\, T^0_k \overset{[\text{on } \Delta^0_k]}{=} v^0_k\, G^0_{B,k}.$$

Since G^0 satisfies assumption (A2), there exists ϱ_k^0, such that

$$v_k^0 = \varrho_k^0\, z_{B,k}^0 = \varrho_k^0\, u_k^0. \tag{34}$$

We now consider $\ell \geq 1$. Using equation (14) we obtain

$$\begin{aligned} 0 = w\,\mathcal{T} &\overset{[\text{on } \Delta^\ell]}{=} v^0\,T^0 + \cdots + v^\ell\,T^\ell \\ &\overset{[\text{on } \Delta^\ell]}{=} \sum_{p=0}^{\ell-1} v^p\, S_{BA}^{p+1} \Bigg(\prod_{r=p+2}^{\ell} S_{AA}^r \Bigg) G_A^\ell + v^\ell\, G_B^\ell. \end{aligned}$$

By induction, all sub-vectors v_k^p, $p < \ell$, can be written as $v_k^p = \varrho_k^p\, z_{B,k}^p = \varrho_k^p\, u_k^p$ whenever Δ_k^p is an orphan, and $v_k^p = \mathbb{O}^T$ otherwise. Using assumption (A1) we obtain

$$v^p\, S_{BA}^{p+1} \Bigg(\prod_{r=p+2}^{\ell} S_{AA}^r \Bigg) = \mathbb{O}_A^\ell \quad \text{for } p = 1, \ldots, \ell-1.$$

For each G^ℓ-connected component Δ_k^ℓ we get

$$0 \overset{[\text{on } \Delta_k^\ell]}{=} \mathbb{O}_{A,k}^\ell\, G_{A,k}^\ell + v_k^\ell\, G_{B,k}^\ell.$$

If Δ_k^ℓ is not an orphan, then we conclude $v_k^\ell = 0$, since the only non-trivial representation of a null function with respect to G^ℓ involves *all* functions (we assumed all elements of z^ℓ to be non-zero).

Otherwise, if Δ_k^ℓ is an orphan, then $G_{A,k}^\ell$ is void. According to assumption (A2) there exists a scalar ϱ_k^ℓ, such that

$$v_k^\ell = \varrho_k^\ell\, z_{B,k}^\ell = \varrho_k^\ell\, u_k^\ell.$$

We conclude that any linear dependency relation can be obtained as a linear combination of vectors w_K^L, where C_K^L is an orphan,

$$w = \sum_{\substack{K,L\,:\,C_K^L \text{ is} \\ \text{an orphan}}} \varrho_K^L\, w_K^L.$$

□

Theorem 25 gives us all linear dependency relations in the truncated hierarchical generating system for certain classes of hierarchies of generating systems, including ZP elements. Consequently, one can restore linear independence by omitting a suitable subset of $\mathcal{T}$. This observation will be used in the next section.

The authors believe that the approach to restore linear independence can be extended to generating systems that possess linear dependency relations with similar properties, such as other box splines on the criss-cross grid. It would be also of interest to analyze the general case of generating systems with possibly unstructured linear dependencies. A more detailed discussion is beyond the scope of the present paper.

7. Adaptive surface fitting with hierarchical ZP elements

Zwart-Powell elements are a particularly interesting class of box splines on the criss-cross grid, since they combine low degree with C^1 smoothness. Based on the theoretical results presented so far we use their truncated hierarchical version to perform surface fitting with adaptive refinement.

7.1. Hierarchical refinement

Given a set of data (points (x_i, y_i) with associated values f_i, $i = 1, \ldots, m$, for a large number of points m), we iteratively compute a least-squares approximation and perform a refinement procedure until the approximation error becomes smaller than a given tolerance.

In our examples, the points are distributed uniformly in the domain Ω (which is an axis-aligned box) and their number is much larger than the dimension of span $\mathcal{T}$. In order to simplify the presentation we do not consider any regularization strategy here, which might be needed in order to avoid problems with singular linear systems[3]. The discussion of such a strategy is beyond the scope of the present paper. Instead, we focus on the issue of adaptive refinement and the elimination of the linear dependencies in the truncated hierarchical generating system.

We consider two different refinement strategies; they will be referred to as *relative* and *full refinement*. Moreover, since the computation of the least-squares approximation needs a generating system that is a basis, we have to find linear dependencies in $\mathcal{T}$ and eliminate suitable functions to restore linear independence. Except for this part (step (b) below), our algorithm is similar to the one used by Giannelli et al. (2012):

Hierarchical approximation.

- *Input:* an initial domain $\Omega \subset \mathbb{R}^2$ together with the data set (x_i, y_i, f_i), $i = 1, \ldots, m$, which associates real values f_i to a set of points $(x_i, y_i) \in \Omega$, a certain tolerance ε, and a maximum value for the number of iteration steps $s_{\max}$.
- *Algorithm:*
  - (a) define an initial generating system G^0 of ZP elements on Ω and initialize $s = 0$;
  - (b) determine the linear dependency relations in the current truncated hierarchical generating system and restore linear independence by omitting suitably chosen functions from $\mathcal{T}$; this creates a modified generating system $\mathcal{T}^\star$.
  - (c) compute the least squares approximation $f = \sum_{\varphi \in \mathcal{T}^\star} c_\varphi \varphi$ that minimizes

$$\sum_{i=0}^{m} (f_i - f(x_i, y_i))^2 \tag{35}$$

[3]For instance, such problems would be present if the number of degrees of freedom in certain subset of Ω exceeds the number of data points in that subset.

with respect to the coefficients of $\mathcal{T}^\star$;

(d) if $\max|f_i - f(x_i, y_i)| > \varepsilon$ and $s < s_{\max}$, then set $s = s + 1$ and apply the hierarchical refinement according to the considered refinement strategy; repeat steps (b) and (c); otherwise the algorithm terminates.

- *Output:* The least-squares approximation f expressed in terms of truncated hierarchical generating system of ZP elements so that $\max|f_i - f(x_i, y_i)| \leq \varepsilon$ or $s = s_{\max}$.

In step (b) we have to consider the connectivity graph at each level of the current truncated hierarchical generating system. We have to find all the connected components of level ℓ that are identified as orphans. Once we find all such components, we can restore linear independence by eliminating one function from each orphan, see Lemma 24 and Theorem 25. These functions are indicated by the circles in the hierarchical meshes shown in the figures that illustrate the examples below.

In step (d) we use two different refinement strategies. Both fit into the same *black-box* procedure.

Hierarchical refinement.

- *Input:* the current hierarchical sequence of nested hierarchical domains together with points (x_i, y_i) marked for refinement;

- *Output:* a nested sequence of domains $\hat{\Omega} \supseteq \cdots \supseteq \hat{\Omega}^N$, so that

$$\hat{\Omega} = \Omega, \quad \hat{\Omega}^\ell \supseteq \Omega^\ell, \quad \ell = 1, \ldots, N, \tag{36}$$

  and the corresponding truncated hierarchical generating system. For each point that is marked for refinement, the level of the domain containing it

$$\max_\ell \{\ell \mid (x_i, y_i) \in \Omega^\ell\}$$

  increases.

The difference between *relative* and *absolute* refinement strategy is the number of points that are marked for refinement. In the case of relative refinement we select the (user-specified) percentage of the points where the largest errors occur. For absolute refinement we simply choose all points where the error exceeds the given threshold.

The enlargement of the domains in (36) is obtained by adding a suitable number of cells from the neighborhood of each point that is marked for refinement to the corresponding domain. This neighborhood is chosen such that at least one ZP element is added to the corresponding generating system. The domain enlargement is followed by a cleaning step in order to restore the property that the domains are nested in the hierarchy.

7.2. Adaptive fitting with relative refinement strategy

We compute the least-squares approximations of the function

$$f(x,y) = \frac{2}{3\exp\left(\sqrt{(3x-5)^2+(3y-5)^2}\right)} + \frac{2}{3\exp\left(\sqrt{(3x+5)^2+(3y+5)^2}\right)} + \frac{2}{3\exp\left(\sqrt{(2x)^2+(2y)^2}\right)}, \quad (x,y) \in [-3,3]^2,$$

shown in Figure 12(a). A similar function was used by Giannelli et al. (2012). We obtained the data by sampling on a uniform grid of size 100×100. In step (d) we performed hierarchical refinement by considering 5% of the points. The tolerance was chosen as $\varepsilon = 10^{-5}$ and we considered 5 steps of refinement. The relative refinement strategy results in the loss of symmetry as it is evident from Figure 12(f). Due to the loss of C^1 smoothness around the 3 "peaks", the refinement is concentrated around them. Table 1 provides information concerning the number of degrees of freedom and the resulting error in the five steps of refinement. The error is measured between the fitted surface and the sampled data, that is, on the 100×100 grid.

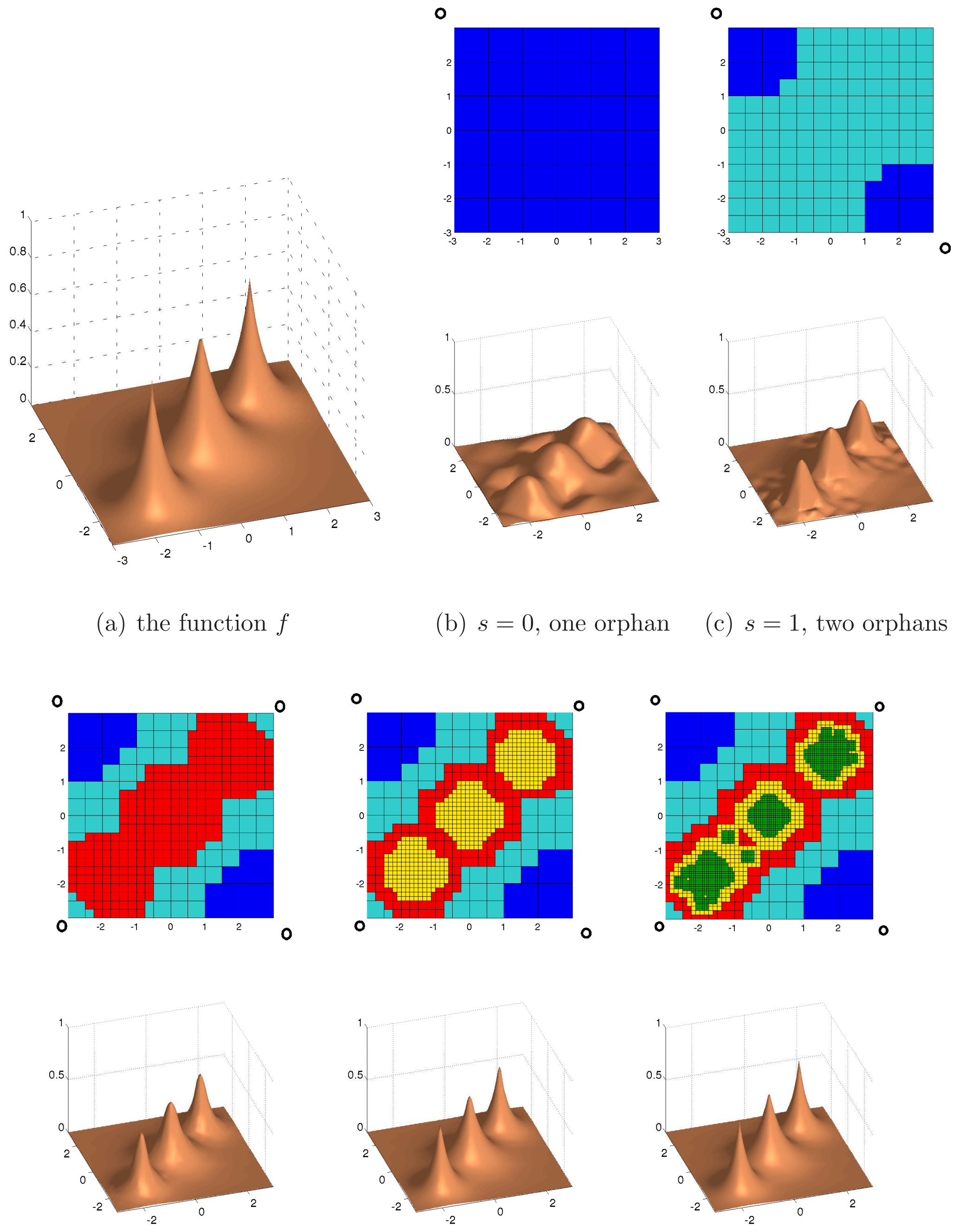
step	dim	ldr	max error
0	63	1	0.41801
1	179	2	0.24848
2	350	4	0.13247
3	662	4	0.06110
4	1490	4	0.00119

Table 1: Using the relative refinement strategy (Section 7.2): The dimension of the generating system after elimination of chosen functions, number of linear dependency relations (ldr) and the maximum error on the 100×100 grid.

In the initial configuration ($s = 0$), the ZP elements possess one linear dependency relation. We recover the linear independence by eliminating the function represented by a circle in the upper left corner in Figure 12(b). After the first refinement, the domain Ω^1 splits Ω into two orphans. Thus, we eliminate two functions from level 0, see Figure 12(c).

Note that we always eliminated functions that are located close to the boundary of the domain Ω. Consequently, the circles representing them are placed outside of Ω, since they are located at the center of the full support (not trimmed by Ω).

Similarly we proceed with the refinement and eliminate four functions after each of the three following refinement steps. In the last step one should note that the connected component of Δ^2 (red), which is surrounded by Δ^3 (yellow), is not an orphan, since it is G^2-connected to the other parts of Δ^2. Similarly, the small yellow spots in the mesh obtained in the last step do not represent additional orphans.



(a) the function f (b) $s = 0$, one orphan (c) $s = 1$, two orphans

(d) $s = 2$, four orphans (e) $s = 3$, four orphans (f) $s = 4$, four orphans

Figure 12: Using the relative refinement strategy (Section 7.2): Adaptively refined meshes and corresponding least-squares approximations of the function in (a) by truncated hierarchical ZP elements; circles denote the functions that were eliminated in order to restore the linear independence of the truncated hierarchical generating system.

7.3. Adaptive fitting with absolute refinement strategy

In the second example we considered the function (also used by Giannelli et al., 2012),

$$f(x,y) = \begin{cases} 1, & y - x \geq 1/2, \\ 2(y-x), & 0 \leq y - x \leq 1/2, \\ 1/2\cos\left(4\pi\sqrt{q(x,y)}\right) + 1/2, & q(x,y) \leq 1/16, \\ 0, & \text{otherwise}, \end{cases} \qquad (\text{for } (x,y) \in [0,2]\times[0,1])$$

where $q(x,y) = (x-3/2)^2 + (y-1/2)^2$. Again we sampled 100×100 points on a uniform grid. The threshold is now $\varepsilon = 10^{-2}$. The results of the surface approximation are summarized in Table 2. After four steps, the error satisfied the given tolerance and no further refinement was necessary.

step	dim	ldr	max error
0	59	1	0.47043
1	179	1	0.03781
2	440	2	0.01667
3	680	3	0.00828

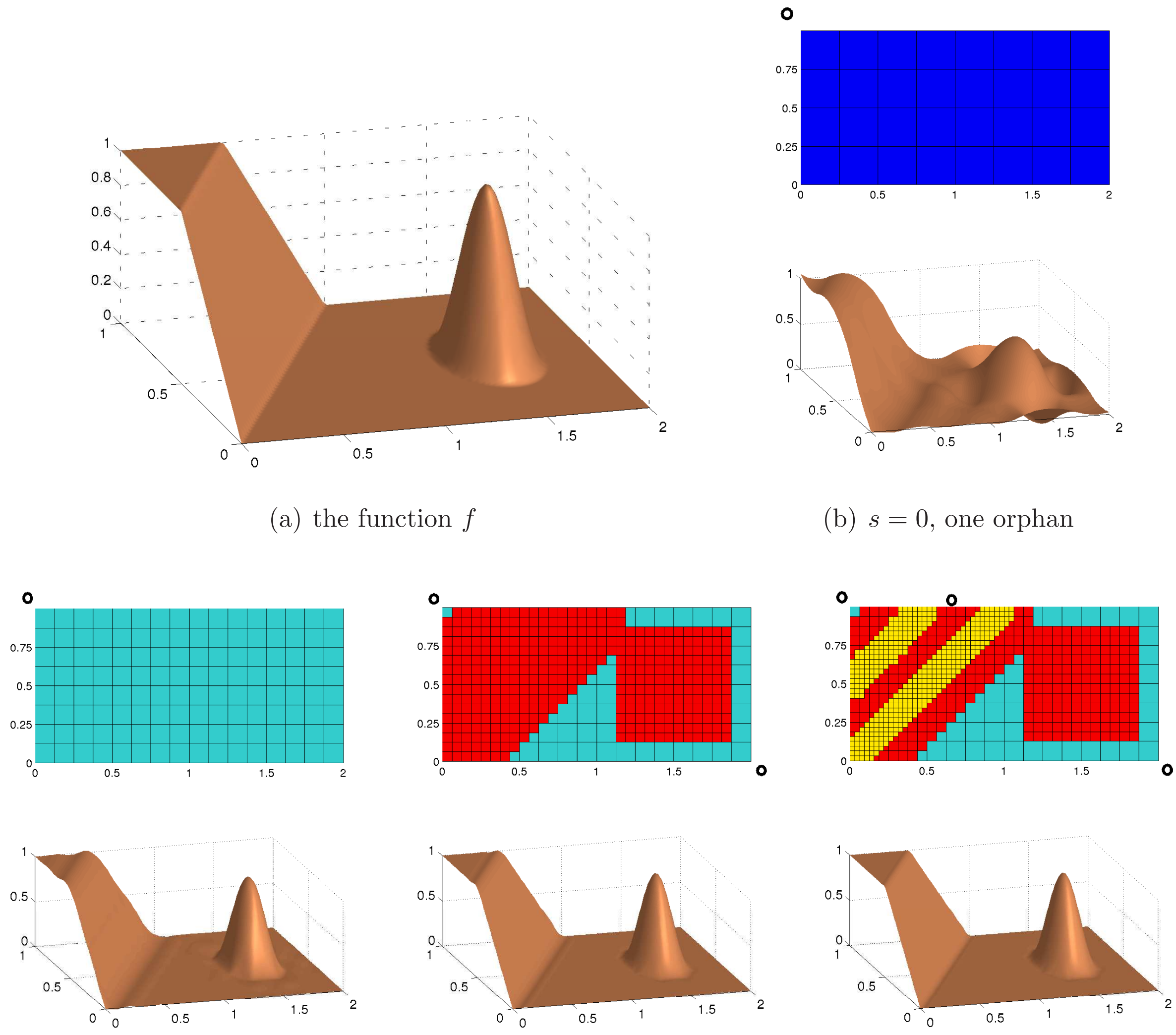
Table 2: Using the absolute refinement strategy (Section 7.3): The dimension of the generating system after elimination of chosen functions, number of linear dependency relations (ldr) and the maximum error on the 100×100 grid.

Once more, the ZP elements of the initial configuration possess one linear dependency relation. It is eliminated by omitting function in the upper left corner, see Fig. 13(b). In this particular example, the dimension of $\mathcal{L}(\mathcal{T})$ increases by one in each step of refinement, and thus, we omit more and more (suitably chosen) functions.

According to our experience, the second refinement strategy performs better, since it better preserves the symmetry that may be present in the data and it also reduces the error faster than the first one. However, if the error threshold is chosen too small (such that it cannot be reached by using only the maximum number of levels that is specified), then the second strategy may lead to an almost global refinement, and the relative refinement strategy would give better results.

8. Summary

We extend the framework of truncated hierarchical bases (Giannelli et al., 2012, 2013) to the case of generating systems, where linear dependencies are possible. We formulated a new framework in order to generalize the existing approach to the new setting. It was shown that many of the advantageous properties of truncated hierarchical basis are also



(a) the function f (b) $s = 0$, one orphan

(c) $s = 1$, one orphan (d) $s = 2$, two orphans (e) $s = 3$, three orphans

Figure 13: Using the absolute refinement strategy (Section 7.3): Adaptively refined meshes and corresponding least-squares approximations of the function in (a) by truncated hierarchical ZP elements; the circles denote the functions that were eliminated in order to restore the linear independence of the truncated hierarchical generating system.

present in the new setting. These include the preservation of coefficients, the nested nature of refinement via domain enlargement, and the partition of unity property.

As specific applications we considered hierarchies of hierarchical B-splines and hierarchical Zwart-Powell (ZP) elements, where we performed adaptive surface fitting. This naturally led us to the question how to eliminate the linear dependencies that are inherited by the hierarchical generating system. For ZP elements, and for the class of hierarchical generating systems with similar properties, we obtained a satisfactory answer.

Our future work will be devoted to further applications of the presented theoretical framework. These may include spline surfaces of general topology, general subdivision surfaces, and more general classes of functions with more complicated linear dependency relations.

Acknowledgements

The authors have been supported by the European Union through the FP7-grant ITN (Marie Curie Initial Training Networks) INSIST (Integrating Numerical Simulation and Geometric Design Technology), GA no. 289361.

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