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Algebraic Differential Equations — Rational Solutions and Beyond

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Abstract. An algebraic ordinary differential equation (AODE) is a polynomial relation between the unknown function and its derivatives. This polynomial defines an algebraic hypersurface, the solution surface. Here we consider AODEs of order 1. From rational parametrizations of the solution surface, we can decide the rational solvability of the given AODE, and in fact compute the general rational solution. This method depends crucially on curve and surface parametrization and the determination of rational invariant algebraic curves.

Transforming the ambient space by some group of transformations, we get a classification of AODEs, such that equivalent equations share the property of rational solvability. In particular we discuss affine and birational transformation groups.

We also discuss the extension of this method to non-rational parametrizations and solutions.

Keywords: Algebraic differential equations; Rational solutions; Symbolic computation.

1. Introduction – The Problem

An *algebraic ordinary differential equation (AODE)* is given by

$$F(x, y, y', \ldots, y^{(n)}) = 0 \ ,$$

where F is a differential polynomial in $K[x]\{y\}$ with K being a differential field and the derivation $'$ being $\frac{d}{dx}$. Such an AODE is *autonomous* if $F \in K\{y\}$.

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The radical differential ideal $\{F\}$ can be decomposed as

$$\{F\} \;=\; \underbrace{(\{F\} : S)}_{general\ component} \;\cap\; \underbrace{\{F, S\}}_{singular\ component} \;,$$

where S is the separant of F; i.e., the derivative of F w.r.t. $y^{(n)}$ (cf. [11]). If F is irreducible, then $\{F\} : S$ is a prime differential ideal, and therefore it has a generic zero. This generic zero is called a *general solution* of the AODE $F(x, y, y', \ldots, y^{(n)}) = 0$. Gröbner basis methods for implicit descriptions of general solutions are discussed in [5].

There are no general methods for determining symbolic solutions of AODEs. We present a method which allows to decide whether a given AODE of order 1 has a rational solution, and if so, will produce all of them. Transforming the ambient space by some group of transformations, we get a classification of AODEs, such that equivalent equations share the property of rational solvability. In particular we discuss affine and birational transformation groups. We also hint at a generalization of this method towards the construction of radical and even transcendental symbolic solutions.

Here we consider AODEs of order 1. More precisely, we deal with the following problem:

Problem AODE1-RatSol

given: an AODE $F(x, y, y') = 0$, F irreducible in $\overline{\mathbb{Q}}[x, y, y']$
decide: does this AODE have a non-singular rational solution ?
find: if so, find all of them; i.e., find the rational general solution.

Example 1.1. [9] Consider the AODE of order 1 defined by

$$F \equiv y'^2 + 3y' - 2y - 3x = 0 \;.$$

This AODE indeed has non-singular rational solutions, and the rational general solution is $y = \frac{1}{2}((x + c)^2 + 3c)$, where c is an arbitrary constant.

The separant of F is $S = 2y' + 3$. So the singular solution of F is $y = -\frac{3}{2}x - \frac{9}{8}$.

In the sequel we will explain how to arrive at this general solution.

So an AODE of order 1 is defined by a polynomial relation $F(x, y, y')$ between x, y, and y'. Neglecting the differential aspect of the problem, we arrive at a surface in 3-space, defined by $F(x, y, z) = 0$. Every rational solution of the problem is a rational curve $(x, y(x), y'(x))$ on this *solution surface*. We divide the problem into two phases: first we decide whether the solution surface contains rational curves, and then we try to find a rational curve also satisfying the differential constraint. For this reason let us briefly recall some relevant facts about rational parametrizations of hypersurfaces.

2. Rational Parametrizations

An *algebraic variety* $\mathcal{V}$ is the zero locus of a (finite) set of polynomials F, or of the ideal $I = \langle F \rangle$. A *rational parametrization* of $\mathcal{V}$ is a rational map $\mathcal{P}$ from a full (affine, projective) space covering $\mathcal{V}$; i.e. $\mathcal{V} = \overline{\mathrm{im}(\mathcal{P})}$, the Zariski closure of the image of $\mathcal{P}$. A variety having a rational parametrization is called *unirational*; and *rational* if $\mathcal{P}$ has a rational inverse.

Example 2.1. The singular cubic $y^2 - x^3 - x^2 = 0$ has the rational, in fact polynomial, parametrization $x(t) = t^2 - 1$, $y(t) = t^3 - t$. This parametrization can be inverted by $t = y/x$. So this singular cubic is a rational curve.

A rational parametrization of a variety is a *generic point* or *generic zero* of the variety; i.e. a polynomial vanishes on the variety if and only if it vanishes on this generic point. Only irreducible varieties can be rational. A rationally invertible parametrization $\mathcal{P}$ is called a *proper* parametrization. Every rational curve or surface has a proper parametrization (theorem of Lüroth, Castelnuovo); but not so in higher dimensions. From the degree of a defining polynomial of a curve we can give a precise degree of a proper parametrization. This degree bound will be essential in the algorithm for determining rational solutions of the corresponding AODE.

Theorem 2.2. [12, Theorem 5] *Let $\mathcal{C}$ be a rational affine curve defined by $f(x, y) = 0$, and let $\mathcal{P}(t) = (p_1(t), p_2(t))$ be a rational parametrization of $\mathcal{C}$. Then $\mathcal{P}(t)$ is proper if and only if* $\deg(\mathcal{P}) = \max\{\deg_x(f), \deg_y(f)\}$. *Furthermore, if $\mathcal{P}$ is proper, then* $\deg(p_1) = \deg_y(f)$, *and* $\deg(p_2) = \deg_x(f)$.

If we know a proper parametrization, we can get any other parametrization by applying a rational map. Proper parametrizations can be transformed into each other by rational maps of degree 1. For details on parametrizations of algebraic curves we refer to [13].

3. Construction of Rational Solutions

Let us first consider the autonomous case, i.e., an AODE of the form $F(y, y') = 0$. A rational solution of $F(y, y') = 0$ corresponds to a rational parametrization of the algebraic curve $F(y, z) = 0$. Indeed, if $y = y(x)$ is a rational solution of the AODE, then $(y(x), y'(x))$ is a rational parametrization of the curve defined by $F(y, z) = 0$. Using the strict degree bounds for proper parametrizations of curves developed in [12] (see Theorem 1), one can show that a rational solution of the AODE determines a proper parametrization of the corresponding curve.

Conversely, from a proper rational parametrization $(y(x), z(x))$ of the curve $F(y, z) = 0$ we get a rational solution of $F(y, y') = 0$ if and only if there is a linear rational function $T(x)$ such that $y(T(x))' = z(T(x))$. If such a $T(x)$ exists, then a rational solution of $F(y, y') = 0$ is given by $y = f(T(x))$. The rational general solution of $F(y, y') = 0$ is $y = f(T(x + c))$, where c is an arbitrary constant.

This idea for deciding the existance of rational solutions, and in the positive case determining the rational general solution, has been developed by R. Feng and X.-S. Gao in [2, 3].

But what if our given AODE is non-autonomous? Can we modify the method so that it also applies to the general case $F(x, y, y') = 0$? It is now natural to assume that the solution surface $F(x, y, z) = 0$ is a rational algebraic surface, i.e. rationally parametrized by $\mathcal{P}(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t))$. Then $\mathcal{P}(s, t)$ creates a rational solution of $F(x, y, y') = 0$ if and only if we can find two rational functions $s(x)$ and $t(x)$ which solve the following *associated system*:

$$s' = \frac{f_1(s,t)}{g(s,t)}, \quad t' = \frac{f_2(s,t)}{g(s,t)}, \tag{1}$$

where $f_1(s, t), f_2(s, t), g(s, t)$ are rational functions in s, t and defined by

$$\begin{aligned}
f_1(s,t) &= \frac{\partial \chi_2(s,t)}{\partial t} - \chi_3(s,t) \cdot \frac{\partial \chi_1(s,t)}{\partial t}, \\
f_2(s,t) &= \chi_3(s,t) \cdot \frac{\partial \chi_1(s,t)}{\partial s} - \frac{\partial \chi_2(s,t)}{\partial s}, \\
g(s,t) &= \frac{\partial \chi_1(s,t)}{\partial s} \cdot \frac{\partial \chi_2(s,t)}{\partial t} - \frac{\partial \chi_1(s,t)}{\partial t} \cdot \frac{\partial \chi_2(s,t)}{\partial s}.
\end{aligned}$$

The construction of the associated system and the construction of rational solutions via this associated system can be found in [6, 7].

Theorem 3.1. *There is a one-to-one correspondence between rational solutions of the algebraic differential equation $F(x, y, y') = 0$, which is parametrized by $\mathcal{P}(s, t)$, and rational solutions of its associated system with respect to $\mathcal{P}(s, t)$. The analogous statement holds for rational general solutions.*

The associated system (1) has the following useful properties:

(i) it is of order 1, so the order is not worse than the order of the original AODE,

(ii) it is of degree 1 in the derivatives of the parameters, whereas the original AODE could be of any degree in the derivative y',

(iii) but most importantly it is autonomous, whereas the original AODE was non-autonomous.

Such systems of algebraic differential equations have been well studied. The important concept here are so-called **invariant algebraic curves**, i.e., curves $G(s, t) = 0$ satisfying

$$G_s \cdot f_1 \cdot g + G_t \cdot f_2 \cdot g \in \langle G \rangle \ , \tag{2}$$

where G_s, G_t denote the derivatives of G w.r.t. s and t, respectively.

Theorem 3.2. *Every non-trivial rational solution of the associated system* (1) *corresponds to a rational invariant algebraic curve* $G(s,t) = 0$ *of this system.*

In the generic case in which the system (1) has no dicritical singularities, there is an upper bound for the degree of irreducible invariant algebraic curves given in [1]. So these invariant curves can be determined.

Finally, as in the autonomous case, we have to decide whether the candidates lead to a solution of the differential problem. I.e., whether we can find a linear transformation to a solution curve.

Theorem 3.3. *Let* $G(s,t) = 0$ *be a rational invariant algebraic curve of the associated system* (1) *such that* $G \nmid g$. *Let* $(s(x), t(x))$ *be a proper rational parametrization of* $G(s,t) = 0$. *W.l.o.g. assume* $s'(x) \neq 0$.

Then $(s(x), t(x))$ *creates a rational solution of the associated system if and only if there is a linear rational function* $T(x)$ *such that*

$$T' = \frac{1}{s'(T)} \cdot \frac{N_1(s(T), t(T))}{M_1(s(T), t(T))}.$$

In this case, $(s(T(x)), t(T(x)))$ *is a rational solution of the associated system.*

Invariant algebraic curves may come in families depending on parameters. Such families give rise to rational general solutions.

Theorem 3.4. *Let* $\mathcal{R}(x) = (s(x), t(x))$ *be a non-trivial rational solution of the system* (1). *Let* $H(s,t)$ *be the monic defining polynomial of the curve parametrized by* $\mathcal{R}(x)$.

Then $\mathcal{R}(x)$ *is a rational general solution of the system* (1) *if and only if the coefficients of* $H(s,t)$ *contain a transcendental constant.*

So now we are ready for presenting an algorithm for deciding whether a given AODE of order 1 has a rational solution, and in the positive case determining the general rational solution.

Algorithm RATSOLVE
Input: a parametrizable AODE of order 1 $F(x, y, y') = 0$;
Output: a rational general solution, if there is one.

1. Compute a proper rational parametrization $\mathcal{P}(s,t) = (\chi_1, \chi_2)$ of the solution surface $F(x, y, z) = 0$;
2. Compute the associated system w.r.t $\mathcal{P}(s,t)$;
3. Compute the set $\mathcal{I}$ of irreducible invariant algebraic curves of the associated system;

4. If $\mathcal{I}$ contains an irreducible invariant algebraic curve $G(s,t) = 0$ with a transcendental coefficient, then check whether $G(s,t) = 0$ is a rational curve;
5. If $G(s,t)$ is a rational curve, then parametrize this curve to find a rational general solution $(s(x), t(x))$ of the system; compare Theorem 4;
6. Compute $c = \chi_1(s(x), t(x)) - x$;
7. Return $y = \chi_2(s(x-c), t(x-c))$.

Example 3.5. [8] Let us come back to Example 1.1 in Section 1, and consider again the differential equation

$$F(x,y,y') \equiv y'^2 + 3y' - 2y - 3x = 0 \ .$$

The solution surface $z^2 + 3z - 2y - 3x = 0$ has the parametrization

$$\mathcal{P}(s,t) = \left(\frac{2s + ts + t^2}{x^2},\ -\frac{3x + t^2}{s^2},\ \frac{t}{s} \right) \ .$$

This is a proper parametrization and its associated system is

$$s' = st, \quad t' = s + t^2 \ .$$

Irreducible invariant algebraic curves of the system are:

$$G(s,t) = s, \quad G(s,t) = t^2 + 2s, \quad G(s,t) = s^2 + ct^2 + 2cs \ .$$

The third invariant algebraic curve, $s^2 + ct^2 + 2cs = 0$, depends on a transcendental parameter c. It can be parametrized by

$$\mathcal{Q}(x) = \left(-\frac{2c}{1 + cx^2},\ -\frac{2cx}{1 + cx^2} \right) \ .$$

Running Step 5 in RATSolve, the differential equation defining the reparametrization is $T' = 1$. Hence $T(x) = x$. So the rational solution in this case is

$$s(x) = -\frac{2c}{1 + cx^2}, \quad t(x) = -\frac{2cx}{1 + cx^2} \ .$$

Since $G(s,t)$ contains a transcendental constant, the above solution is a rational general solution of the associated system. Therefore, the rational general solution of $F(x,y,y') = 0$ is $y = \frac{1}{2}x^2 + \frac{1}{c}x + \frac{1}{2c^2} + \frac{3}{2c}$, which, after a change of parameter, can be written as $y = \frac{1}{2}(x^2 + 2cx + c^2 + 3c)$.

4. Geometrical Classification of AODEs

This geometrical treatment of algebraic differential equations suggests the consideration of groups of transformations leaving the associated system of an

AODE invariant. Orbits w.r.t. such a transformation group contain AODEs of equal complexity in terms of determining rational solutions. We study orbits w.r.t affine and birational transformations. It turns out that being autonomous is not an intrinsic property of an AODE.

The group $\mathcal{G}_a$ of affine transformations

$$\begin{array}{rcl} L : \mathbb{A}^3(K) & \longrightarrow & \mathbb{A}^3(K) \\ v & \mapsto & \begin{pmatrix} 1 & 0 & 0 \\ b & a & 0 \\ 0 & 0 & a \end{pmatrix} v + \begin{pmatrix} 0 \\ c \\ b \end{pmatrix} \end{array},$$

$a \neq 0$, leaves the associated system of an AODE invariant, and therefore also the rational solvability; cf. [9]. The group $\mathcal{G}_a$ defines a group action on AODEs by

$$\begin{array}{rcl} \mathcal{G}_a \times \mathcal{AODE} & \rightarrow & \mathcal{AODE} \\ (L, F) & \mapsto & L \cdot F = (F \circ L^{-1})(x, y, y') \ . \end{array}$$

Theorem 4.1. *Let F be a parametrizable AODE, and $L \in \mathcal{G}_a$. For every proper rational parametrization $\mathcal{P}$ of the surface $F(x, y, z) = 0$, the associated system of $F(x, y, y') = 0$ w.r.t. $\mathcal{P}$ and the associated system of $(L \cdot F)(x, y, y') = 0$ w.r.t. $L \circ \mathcal{P}$ are equal.*

Example 4.2. Again we consider the differential equation

$$F(x, y, y') \equiv y'^2 + 3y' - 2y - 3x = 0 \ .$$

We first check whether in the class of F w.r.t. affine transformations contains an autonomous AODE. For this purpose, we apply a generic transformation L to F to get

$$(L \cdot F)(x, y, y') = \frac{1}{a^2} y'^2 + \frac{3}{a} y' - \frac{2b}{a^2} y' - \frac{2}{a} y + \frac{2b}{a} x - 3x - \frac{3b}{a} + \frac{b^2}{a^2} + \frac{2c}{a}.$$

Therefore, for every $a \neq 0$ and b such that $2b - 3a = 0$, we get an autonomous AODE. In particular, for $a = 1$, $b = 3/2$, and $c = 0$ we get

$$L = \left[\begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{3}{2} \end{pmatrix} \right],$$

i.e., we obtain

$$F(L^{-1}(x, y, y')) \equiv y'^2 - 2y - \frac{9}{4} = 0 \ .$$

Let us classify some well-known forms of differential equations w.r.t. affine transformations:

(i) *Equations solvable for* y: invariant w.r.t. affine transformations
equation: $y = G(x, y')$ for a rational G
proper parametrization of solution surface: $(s,\ G(s,t),\ t)$
associated system: $s' = 1, \quad t' = (t - G_s(s,t))/G_t(s,t)$

(ii) *Equations solvable for* y': invariant w.r.t. affine transformations
equation: $y' = G(x, y)$ for a rational G
proper parametrization of solution surface: $(s,\ t,\ G(s,t))$
associated system: $s' = 1, \quad t' = G(s,t)$

(iii) *Equations solvable for* x: *not* invariant w.r.t. affine transf.
equation: $x = G(y, y')$ for a rational G
proper parametrization of solution surface: $(G(s,t),\ s,\ t)$
associated system: $s' = t, \quad t' = (1 - t \cdot G_s(s,t))/G_t(s,t)$

So by a transformation in $\mathcal{G}_a$ we might transform an equation not solvable for x into an equation solvable for x and thus make it amenable to well-known solution methods.

Now we turn our attention to a wider class of geometrical transformations, namely birational transformations; cf. [10]. The group $\mathcal{G}_b$ of birational transformations from K^3 to K^3 of the form

$$\Phi(u_1, u_2, u_3) \;=\; \left(u_1,\quad \frac{au_2 + b}{cu_2 + d},\quad \frac{\partial}{\partial u_1}\left(\frac{au_2 + b}{cu_2 + d}\right) + \frac{\partial}{\partial u_2}\left(\frac{au_2 + b}{cu_2 + d}\right) \cdot u_3\right) ,$$

where $a, b, c, d \in K[u_1]$ such that $ad - bc \neq 0$, defines a group action on $\mathcal{AODE}$ by $\Phi \cdot F = (F \circ \Phi^{-1})(x, y, y')$. These birational transformations leave the associated system of an AODE invariant, and therefore also the rational solvability.

Example 4.3. Consider the first order AODE

$$\begin{aligned} & F(x, y, y') \\ =\; & 25x^2y'^2 - 50xyy' + 25y^2 + 12y^4 - 76xy^3 + 168x^2y^2 - 144x^3y + 32x^4 \\ =\; & 0. \end{aligned}$$

Using the transformation

$$\Phi(u, v, w) = \left(u,\ \frac{u - 3v}{-2u + v},\ \frac{-5v}{(2u - v)^2} + \frac{5u}{(2u - v)^2}w\right)$$

we get the autonomous equation

$$G(y, y') = F(\Phi^{-1}(x, y, y')) = y'^2 - 4y = 0.$$

Observe that F cannot be transformed into an autonomous AODE by affine transformations. The rational general solution $y = (x + c)^2$ of $G(y, y') = 0$ is transformed into the rational general solution of $F(x, y, y') = 0$, yielding

$$y = \frac{x(2(x + c)^2 + 1)}{(x + c)^2 + 3}.$$

W.r.t. birational transformations also the class of equations solvable for y is not closed. So even more differential equations can be transformed into well-studied forms.

5. Extension to Non-Rational Solutions

Recently G. Grasegger started to extend these ideas to the problem of computing non-rational solutions of autonomous AODEs. Here we can only hint at his methods; details are given in [4].

Suppose y is a solution of the autonomous AODE $F(y, y') = 0$. Then $P_y = (y(t), y'(t))$ is a parametrization of the solution surface $F(y, z) = 0$. For any parametrization $P = (r(t), s(t))$ of the solution surface we consider $A_P = s(t)/r'(t)$. Assume the parametrization is of the form $P_g = (r(t), s(t)) = (y(g(t)), y'(g(t)))$, for unknown y and g. It turns out that $A_{P_g} = 1/g'$. If we could find g, and its inverse g^{-1}, we also could find y:

$$g'(t) = \frac{1}{A_{P_g}}, \quad g(t) = \int g'(t)dt, \quad y(x) = r(g^{-1}(x)) \ .$$

So we can determine a solution if we can compute the integral of g' and the inverse of g.

Example 5.1. By this approach we can determine the following non-rational solutions of the respective AODEs.

(i) $y^8 y' - y^5 - y' = 0$:
parametrization: $(\frac{1}{t}, \frac{t^3}{1-t^8})$, $\quad g(t) = \frac{1+t^8}{4t^4}$,
radical solution: $y(x) = -\left(2(x+c) - \sqrt{-1 + 4(x+c)^2}\right)^{-1/4}$

(ii) $4y^7 - 4y^5 - y^3 - 2y' - 8y^2y' + 8y^4y' + 8yy'^2 = 0$: (genus 1)
parametrization: $\left(\frac{1}{t}, \frac{-4+4t^2+t^4}{t(4t^2-4t^4-t^6-\sqrt{t^{12}+8t^{10}+16t^8-16t^4})}\right)$
radical solution: $y(x) = -\frac{\sqrt{1+c+x}}{\sqrt{1+(c+x)^2}}$

(iii) $y^3 + y^2 + y'^2 = 0$:
parametrization: $(-1-t^2, t(-1-t^2))$, $\quad g(t) = 2\mathrm{arctan}(t)$,
trigonometric solution: $y(x) = -1 - \tan\left(\frac{x+c}{2}\right)^2$

(iv) $y^2 + y'^2 + 2yy' + y = 0$:
parametrization: $\left(-\frac{1}{(1+t)^2}, -\frac{t}{(1+t)^2}\right)$
$g(t) = -2\log(t) + 2\log(1+t)$
exponential solution: $y(x) = -e^{-x}(-1 + e^{x/2})^2$.

6. Conclusion

Summarizing, let us recollect what we have achieved:

(i) we can decide whether an AODE (autonomous or non-autonomous) of order 1 has rational solutions; and if it has rational solutions, we can determine the general rational solution;

(ii) we have a characterization of the affine and birational transformations of the ambient space leaving the rational solvability of AODEs invariant; this may lead to a simplification of the equation and to a form amenable to the application of well-known solution methods;

(iii) we have described a general method for determining whether an autonomous AODE has a solution in a given class of functions (rational, radical, transcendental); the method depends on the solvability of the problems of integration and inversion in the class of functions; however, this is not (yet) a complete method.

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