Rational General Solutions of Trivariate Rational Systems of Autonomous ODEs

Yanli Huang, L. X. Châu Ngô and Franz Winkler

MACIS 2011

Abstract. We generalize the method of Ngô and Winkler [5] for finding rational general solutions of a plane rational differential system to the case of a trivariate rational differential system. We give necessary and sufficient conditions for the trivariate rational differential system to have a rational solution based on proper reparametrization of invariant algebraic space curves. We also present a criterion for a rational solution to be a rational general solution.

Mathematics Subject Classification (2000). Primary 68W30.

Keywords. Rational general solution, invariant algebraic space curve, proper parametrization.

1. Introduction

In [5], two of the authors presented a method for solving planar rational systems of autonomous ODEs of order 1. In this paper, we are interested in looking for rational general solutions of trivariate rational systems of autonomous ODEs of the form

$$s_i' = \frac{U_i(s_1, s_2, s_3)}{V_i(s_1, s_2, s_3)}, \quad i = 1, 2, 3, \tag{1.1}$$

where $U_i, V_i \in \mathbb{K}[s_1, s_2, s_3]$, $\mathbb{K}$ an algebraically closed field of characteristic zero. The main idea is to generalize the notion of invariant algebraic curve in [5] for planar rational systems to trivariate rational systems. In fact, we define an invariant algebraic space curve in the trivariate case by using the saturation ideal of a regular chain. Rational solutions of the system (1.1) can be found by the technique of reparametrization of the invariant algebraic space curve. In addition, we give a criterion for deciding when a rational solution would be a rational general solution. Note that only irreducible space curves can be parametrizable. So we only consider irreducible invariant algebraic space curves.

A rational solution of the system (1.1) is a 3-tuple of rational functions which satisfies the given system. A solution $(s_1(x), s_2(x), s_3(x))$ is *trivial* if all the $s_i(x)$ are constant. Because trivial rational solutions are easy to be computed, we restrict our attention to the computation of non-trivial rational solution. From now on, we simply write rational solution for it if no confusion can arise.

2. Invariant algebraic space curves

For convenience, we define the differential operator

$$\mathcal{D} = \sum_{j=1}^{3} U_j W_j \frac{\partial}{\partial s_j},$$

where $W_j = \frac{\mathrm{lcm}(V_1,V_2,V_3)}{V_j}$, U_j and V_j are the numerator and denominator of the right side of the system (1.1), respectively. Hence, for any $H \in \mathbb{K}[s_1, s_2, s_3]$, $\mathcal{D}(H) = \sum_{j=1}^{3} U_j W_j H_{s_j}$, where H_{s_j} is the partial derivative of H w.r.t. s_j. Let $\mathbb{T}$ be any triangular set in $\mathbb{K}[s_1, s_2, s_3]$, the *saturation* of $\mathbb{T}$ is the ideal

$$\mathrm{sat}(\mathbb{T}) := \{G \in \mathbb{K}[s_1, s_2, s_3] \mid H^q G \in \langle \mathbb{T} \rangle \text{ for some } q \in \mathbb{N}\},$$

where H is the product of initials of all polynomials in $\mathbb{T}$. Note that we use the algebraic variable ordering $s_1 < s_2 < s_3$ in the polynomial ring $\mathbb{K}[s_1, s_2, s_3]$.

Definition 2.1. Let $H_1 \in \mathbb{K}[s_1, s_2]$, $H_2 \in \mathbb{K}[s_1, s_2, s_3]$. If $\mathcal{D}(H_1), \mathcal{D}(H_2) \in \mathrm{sat}(H_1, H_2)$, and $\{H_1, H_2\}$ is an irreducible regular chain, then $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$ is an *invariant algebraic space curve* of the system (1.1).

If we choose an upper bound for the degrees of H_1 and H_2, we can make an ansatz for the undetermined coefficients, and determine H_i by solving the corresponding algebraic equations in these coefficients. Then the implicit representation of the invariant algebraic space curve can be given by computing $\mathrm{sat}(H_1, H_2)$ according to necessity. In fact, it has been proved in [4] that $\mathrm{sat}(\mathbb{T}) = \langle \mathbb{T} \rangle$ if and only if $\mathbb{T}$ is primitive for the given regular chain $\mathbb{T}$. They also presented an algorithm for deciding whether $\mathbb{T}$ is primitive. Their experimental results show that this equation holds quite often in practice.

Example. Consider the trivariate polynomial system of autonomous ODEs

$$s_1' = s_1 s_3 - s_2, \quad s_2' = 2s_1^2 - s_1 s_2, \quad s_3' = s_1^2. \tag{2.1}$$

First, we look for an invariant algebraic space curve $\mathbf{Z}(\mathrm{sat}(H_1(s_1, s_2), H_2(s_1, s_2, s_3)))$ satisfying $\deg(H_1) = \deg(H_2) = 1$. We w.l.o.g. assume that the irreducible regular chain $\{H_1, H_2\} = \{s_2 + c_1 s_1 + c_2, s_3 + c_3 s_2 + c_4 s_1 + c_5\}$. Then the Gröbner basis of $\mathrm{sat}(H_1, H_2)$ w.r.t. the lexicographic order determined by $s_1 < s_2 < s_3$ is

$$\mathbb{G} = \{s_2 + c_1 s_1 + c_2, s_3 + (c_4 - c_3 c_1) s_1 + c_5 - c_3 c_2\},$$

and

$$\mathcal{D}(H_1) = c_1 s_3 s_1 - s_2 s_1 - c_1 s_2 + 2s_1^2, \quad \mathcal{D}(H_2) = c_4 s_3 s_1 - c_3 s_2 s_1 - c_4 s_2 + (2c_3 + 1) s_1^2.$$

It follows that

$$\begin{aligned}\operatorname{nform}(\mathcal{D}(H_1),\mathbb{G}) =&(c_1 - c_1c_4 + c_1^2c_3 + 2)s_1^2 + (c_1^2 + c_2 - c_1c_5 + c_1c_2c_3)s_1 + c_1c_2,\\ \operatorname{nform}(\mathcal{D}(H_2),\mathbb{G}) =&(c_1c_3 - c_4^2 + c_1c_3c_4 + 2c_3 + 1)s_1^2 + (c_1c_4 + c_2c_3 - c_4c_5\\ &+ c_2c_3c_4)s_1 + c_2c_4.\end{aligned}$$

Therefore, the algebraic system of equations on the coefficients of H_1 and H_2 is

$$\begin{cases} c_1 - c_1c_4 + c_1^2c_3 + 2 = 0\\ c_1^2 + c_2 - c_1c_5 + c_1c_2c_3 = 0\\ c_1c_2 = 0\\ c_1c_3 - c_4^2 + c_1c_3c_4 + 2c_3 + 1 = 0\\ c_1c_4 + c_2c_3 - c_4c_5 + c_2c_3c_4 = 0\\ c_2c_4 = 0.\end{cases}$$

By solving this system, we obtain the solution

$$\{c_1 = -1, c_2 = 0, c_3 = -1 - c_4, c_4 = c_4, c_5 = -1\}.$$

This gives an invariant algebraic space curve $\mathbf{Z}(\operatorname{sat}(H_1, H_2))$, where

$$H_1 = s_2 - s_1, \quad H_2 = s_3 - (1 + c_4)s_2 + c_4s_1 - 1.$$

Now we ask for an invariant algebraic space curve $\mathbf{Z}(\operatorname{sat}(\tilde{H}_1(s_1, s_2), \tilde{H}_2(s_1, s_2, s_3)))$ such that $\deg(\tilde{H}_1) = 2$, $\deg(\tilde{H}_2) = 1$. Assume that

$$\tilde{H}_1 = s_2 + c_1s_1^2 + c_2s_1 + c_3, \quad \tilde{H}_2 = s_3 + c_4s_2 + c_5s_1 + c_6,$$

then the following solutions are computed by using the same procedure as above

$$\begin{aligned}&\{c_1 = 0, c_2 = -1, c_3 = 0, c_4 = -1 - c_5, c_5 = c_5, c_6 = -1\},\\ &\{c_1 = 3/2, c_2 = -4, c_3 = 0, c_4 = 0, c_5 = 2, c_6 = -4\}.\end{aligned}$$

The first solution corresponds to the computed invariant algebraic space curve $\mathbf{Z}(\operatorname{sat}(H_1, H_2))$, where $\deg(H_1) = \deg(H_2) = 1$. For the second solution, it determines another invariant algebraic space curve $\mathbf{Z}(\operatorname{sat}(\tilde{H}_1, \tilde{H}_2))$, where

$$\tilde{H}_1 = s_2 + \frac{3}{2}s_1^2 - 4s_1, \quad \tilde{H}_2 = s_3 + 2s_1 - 4.$$

In fact, $\mathbf{Z}(\operatorname{sat}(\tilde{H}_1, \tilde{H}_2))$ is the only invariant algebraic space curve satisfying $\deg(\tilde{H}_1) = 2$ and $\deg(\tilde{H}_2) = 1$ in the above example, because the other possible cases of $\tilde{H}_1$ of degree 2 lead to algebraic systems of equations having no solution. Observe that H_i and $\tilde{H}_i$ in Example 2 are monic polynomials, we have $\operatorname{sat}(H_1, H_2) = \langle H_1, H_2\rangle$ and $\operatorname{sat}(\tilde{H}_1, \tilde{H}_2) = \langle \tilde{H}_1, \tilde{H}_2\rangle$. It follows that $\mathbf{Z}(\operatorname{sat}(H_1, H_2)) = \mathbf{Z}(H_1, H_2)$ and $\mathbf{Z}(\operatorname{sat}(\tilde{H}_1, \tilde{H}_2)) = \mathbf{Z}(\tilde{H}_1, \tilde{H}_2)$. Therefore, one of the computed invariant algebraic space curves is a space line determined by the intersection of surfaces $H_1 = 0$ and $H_2 = 0$, the other one is a space conic determined by the intersection of $\tilde{H}_1 = 0$ and $\tilde{H}_2 = 0$.

3. Rational general solutions

In this section, we explain how to find a rational general solution of trivariate rational systems of autonomous ODEs.

Lemma 3.1. *Let* $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$ *be a rational invariant algebraic space curve of the system* (1.1)*, where* $H_1 \in \mathbb{K}[s_1, s_2]$*,* $H_2 \in \mathbb{K}[s_1, s_2, s_3]$*, and* $(s_1(x), s_2(x), s_3(x))$ *is a rational parametrization of* $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$*. If* $V_j(s_1(x), s_2(x), s_3(x)) \neq 0$ *for* $j = 1, 2, 3$*, then*

$$s_1'(x) \cdot \frac{U_k(s_1(x), s_2(x), s_3(x))}{V_k(s_1(x), s_2(x), s_3(x))} = s_k'(x) \cdot \frac{U_1(s_1(x), s_2(x), s_3(x))}{V_1(s_1(x), s_2(x), s_3(x))}, \quad k = 2, 3.$$

The condition $V_j(s_1(x), s_2(x), s_3(x)) \neq 0$ in the above lemma means $V_j \notin \mathrm{sat}(H_1, H_2)$. The parametrization problem for algebraic plane curves has been studied intensively, e.g. in [6, 7, 8]. Therefore the key point for computing a rational parametrization of the invariant algebraic space curve mentioned in Lemma 3.1 is to birationally project it to a plane curve (see [1]). Lemma 3.1 tells us that not every rational parametrization of a rational invariant algebraic space curve can provide a rational solution of the system (1.1). They are the candidates of rational solutions. In the following, we give a theorem which provides necessary and sufficient conditions for the rational system to have a rational solution. Before that, we need to introduce an important property of proper parametrizations of a rational space curve.

Lemma 3.2. *Let* $\mathcal{P}_1(t)$ *be a proper, i.e. rationally invertible, parametrization of an affine rational space curve* $\mathcal{C}$*, and let* $\mathcal{P}_2(t)$ *be any other rational parametrization of* $\mathcal{C}$*.*

(a) *There exists a non-constant rational function* $R(t)$ *such that* $\mathcal{P}_2(t) = \mathcal{P}_1(R(t))$*.*
(b) $\mathcal{P}_2(t)$ *is proper if and only if there exists a linear rational function* $L(t)$ *such that* $\mathcal{P}_2(t) = \mathcal{P}_1(L(t))$*.*

Note that if $(s_1(x), s_2(x), s_3(x))$ is a proper parametrization of the given rational space curve, then at least one of $s_i(x)$ is non-constant. In the following, we present a method for finding a rational solution of the system (1.1) based on proper parametrizations of its invariant algebraic space curve.

Theorem 3.3. *Let* $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$ *be a rational invariant algebraic space curve of the system* (1.1) *such that* $V_j \notin \mathrm{sat}(H_1, H_2)$ *for* $1 \le j \le 3$*, and* $(s_1(x), s_2(x), s_3(x))$ *be an arbitrary proper rational parametrization of the space curve* $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$*. Then the system* (1.1) *has a rational solution*

$$(\widehat{s}_1(x), \widehat{s}_2(x), \widehat{s}_3(x)) = (s_1(T(x)), s_2(T(x)), s_3(T(x)))$$

corresponding to $\mathbf{Z}(\mathrm{sat}(H_1, H_2))$ *if and only if there exists a linear rational transformation* $T(x) = \frac{ax+b}{cx+d}$ *which is a rational solution of one of the following autonomous differential equations*

$$T'(x) = \frac{1}{s_i'(T(x))} \cdot \frac{U_i(s_1(T(x)), s_2(T(x)), s_3(T(x)))}{V_i(s_1(T(x)), s_2(T(x)), s_3(T(x)))}, \text{ when } s_i'(x) \neq 0,\ i = 1, 2, 3. \tag{3.1}$$

Note that the solvability of the differential equation in (3.1) does not depend on the choice of the proper parametrization of the invariant algebraic space curve. Two different rational solutions from two different proper parametrizations of the same invariant algebraic space curve are related to each other by a shifting of the variable.

Example. Continue considering the system (2.1). In fact, it is easy to see that

$$(s_1(x), s_2(x), s_3(x)) = (x, x, x + 1)$$

is a proper parametrization of the invariant algebraic space curve $\mathbf{Z}(H_1, H_2)$. Since $s_1'(x) \neq 0$, by solving the differential equation

$$T'(x) = \frac{s_1(T(x))s_3(T(x)) - s_2(T(x))}{s_1'(T(x))} = T^2(x),$$

we have $T(x) = -\frac{1}{x}$. According to Theorem 3.3,

$$(s_1(T(x)), s_2(T(x)), s_3(T(x))) = (-\frac{1}{x}, -\frac{1}{x}, -\frac{1}{x} + 1)$$

is a rational solution of the system (2.1) corresponding to $\mathbf{Z}(H_1, H_2)$. Similarly,

$$(s_1(x), s_2(x), s_3(x)) = (x, -\frac{3}{2}x^2 + 4x, -2x + 4)$$

is a proper parametrization of the invariant algebraic space curve $\mathbf{Z}(\tilde{H}_1, \tilde{H}_2)$. Note that $s_1'(x) \neq 0$, by solving the differential equation

$$T'(x) = \frac{s_1(T(x))s_3(T(x)) - s_2(T(x))}{s_1'(T(x))} = -\frac{1}{2}T^2(x),$$

we have $T(x) = \frac{2}{x}$. According to Theorem 3.3,

$$(s_1(T(x)), s_2(T(x)), s_3(T(x))) = (\frac{2}{x}, -\frac{6}{x^2} + \frac{8}{x}, -\frac{4}{x} + 4)$$

is a rational solution of the system (2.1) corresponding to $\mathbf{Z}(\tilde{H}_1, \tilde{H}_2)$.

In what follows, we consider the differential polynomial set $\mathbb{A} = \{A_1, A_2, A_3\} \subset \mathbb{K}\{s_1, s_2, s_3\}$, where $A_i = V_i s_i' - U_i$, and U_i, V_i come from the system (1.1). Note that

$$[\mathbb{A}] : S_{\mathbb{A}}^{\infty} = \{G \in \mathbb{K}\{s_1, s_2, s_3\} \mid (\prod_{i=1}^{3} V_i)^q G \in [\mathbb{A}] \text{ for some } q \in \mathbb{N}\}$$

is a prime differential ideal. From this, it can be easily prove that

$$[\mathbb{A}] : S_{\mathbb{A}}^{\infty} = \{\mathbb{A}\} : S_{\mathbb{A}}^{\infty}.$$

According to Proposition 2.1 in [2], we have the following decomposition

$$\{\mathbb{A}\} = ([\mathbb{A}] : S_{\mathbb{A}}^{\infty}) \bigcap (\bigcap_{i=1}^{3} \{\mathbb{A}, V_i\}),$$

where $[\mathbb{A}] : S_{\mathbb{A}}^{\infty}$ and $\bigcap_{i=1}^{3} \{\mathbb{A}, V_i\}$ define the *general component* and the *singular component* of $\mathbb{A}$, respectively. Observe that every solution of the system (1.1) is a zero of $\mathbb{A}$ for which none of the V_i's vanish, i.e, it is a zero of $[\mathbb{A}] : S_{\mathbb{A}}^{\infty}$. Therefore, a general solution of the system (1.1) can be defined as follows.

Definition 3.4. A generic zero of the prime differential ideal $[\mathbb{A}] : S_{\mathbb{A}}^{\infty}$ is said to be a *general solution* of the system (1.1).

Observe that our previous method can compute rational solutions of the system (1.1). The following theorem gives us a criterion for a rational solution to be a rational general solution.

Theorem 3.5. *Suppose that* $(s_1(x), s_2(x), s_3(x))$ *is a rational solution of the system* (1.1)*, which is implicitly represented by the ideal generated by* G_i *with the following form*

$$G_i = A_i - c_i B_i, \quad i = 1, \cdots, m, \tag{3.2}$$

where $A_i, B_i \in \mathbb{K}[s_1, s_2, s_3]$*, and* c_i *are transcendental constants over* $\mathbb{K}$*. Then* $(s_1(x), s_2(x), s_3(x))$ *is a rational general solution of the system* (1.1)*.*

Example. Consider the following trivariate rational system of autonomous ODEs

$$s_1'(x) = 1, \quad s_2'(x) = -\frac{s_3}{s_1^2}, \quad s_3'(x) = \frac{2s_3}{s_1}.$$

By using our method, we can find a rational solution $(x, -c_1x - c_2, c_1x^2)$. It can be implicitly represented by the ideal $\langle -s_3 + c_1s_1^2, s_3 + s_2s_1 + c_2s_1 \rangle$, which has generators of the form in (3.2), where

$$A_1 = -s_3, \quad B_1 = -s_1^2, \quad A_2 = s_3 + s_2s_1, \quad B_2 = -s_1.$$

According to Theorem 3.5, $(x, -c_1x - c_2, c_1x^2)$ is a rational general solution of the given system.

4. Conclusions

In this paper, we have presented a method for finding rational general solutions of trivariate rational differential systems. Our method can be extended to multivariate rational differential systems. In addition, there is a connection between invariant algebraic space curves and rational first integrals. This relationship helps us to study rational general solutions of rational differential systems via rational first integrals and vice versa. Observe that we don't have a degree bound for irreducible invariant algebraic space curves. This is similar to the problem arising in a generalization of Hubert's method [3] to higher order ODEs; namely to determine a bound on the number of derivations to be considered. Therefore, it is interesting to develop some methods for finding invariant algebraic space curves in other representations without involving the degree bound problem, e.g. looking for parametric representations of invariant algebraic space curves directly.

References

[1] S. S. Abhyankar and C. L. Bajaj, *Automatic parametrization of rational curves and surfaces IV: algebraic space curves.* ACM Transactions on Graphics **8**, no. **4** (1989), 325-334.

[2] E. Hubert, *Factorization-free decomposition algorithms in differential algebra.* Journal of Symbolic Computation **29** (2000), 641-662.

[3] E. Hubert, *The general solution of an ordinary differential equation.* In Proceedings of the 1996 International Symposium on Symbolic and Algebraic Computation (ISSAC'1996), 189-195, ACM Press, New York, 1996.

[4] F. Lemaire, M. Moreno Maza, W. Pan and Y. Xie, *When does* $\langle T \rangle$ *equal* $\mathrm{sat}(T)$? In Proceedings of the 2008 International Symposium on Symbolic and Algebraic Computation (ISSAC'2008), 207-214, ACM Press, New York, 2008.

[5] L. X. C. Ngô and F. Winkler, *Rational general solutions of planar rational systems of autonomous ODEs.* Journal of Symbolic Computation **46** (2011), 1173-1186.

[6] J. R. Sendra and F. Winkler, *Parametrization of algebraic curves over optimal field extensions.* Journal of Symbolic Computation **23** (1997), 191-207.

[7] J. R. Sendra and F. Winkler, *Tracing index of rational curve parametrization.* Computer Aided Geometric Design **18** (2001), 771-795.

[8] J. R. Sendra, F. Winkler and S. Pérez-Díaz, *Rational Algebraic Curves: A Computer Algebra Approach.* Springer, Berlin, 2008.

Acknowledgment

The authors would like to thank the anonymous referees for their helpful comments. The work reported here has been supported by the Austrian Science Foundation (FWF) via the Doctoral Program "Computational Mathematics" (W1214), Project DK11, the Project DIFFOP (P20336-N18), the SKLSDE Open Fund SKLSDE-2011KF-02, the NSFC Project 10871017, the Natural Science Foundation of Beijing (1102026), and the China Scholarship Council.

Yanli Huang
LMIB - SKLSDE - School of Mathematics and Systems Science
Beihang University
100191 Beijing
China
Research Institute for Symbolic Computation
Johannes Kepler University Linz
4040, Linz
Austria
e-mail: Yanli.Huang@risc.jku.at

L. X. Châu Ngô
Research Institute for Symbolic Computation
Johannes Kepler University Linz
4040, Linz
Austria
e-mail: Ngo.Chau@risc.jku.at

Franz Winkler
Research Institute for Symbolic Computation
Johannes Kepler University Linz
4040, Linz
Austria
e-mail: Franz.Winkler@risc.jku.at