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A DECOMPOSITION RESULT FOR BIHARMONIC PROBLEMS AND THE HELLAN-HERRMANN-JOHNSON METHOD*

WOLFGANG KRENDL† AND WALTER ZULEHNER‡

Abstract. For the first biharmonic problem a mixed variational formulation is introduced which is equivalent to a standard primal variational formulation on arbitrary polygonal domains. Based on a Helmholtz decomposition for an involved nonstandard Sobolev space it is shown that the biharmonic problem is equivalent to three (consecutively to solve) second-order elliptic problems. Two of them are Poisson problems, the remaining one is a planar linear elasticity problem with Poisson ratio 0. The Hellan-Herrmann-Johnson mixed method and a modified version are discussed within this framework. The unique feature of the proposed solution algorithm for the Hellan-Herrmann-Johnson method is that it is solely based on standard Lagrangian finite element spaces and standard multigrid methods for second-order elliptic problems and it is of optimal complexity.

Key words. biharmonic equation, Hellan-Herrmann-Johnson method, mixed methods, Helmholtz decomposition

AMS subject classifications. 65N22, 65F10, 65N55

1. Introduction. We consider the first biharmonic boundary value problem: For given f, find w such that

$$\Delta^2 w = f \quad \text{in } \Omega, \qquad w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma, \tag{1.1}$$

where Ω is an open and bounded set in $\mathbb{R}^2$ with a polygonal Lipschitz boundary Γ, Δ and $\partial/\partial n$ denote the Laplace operator and the derivative in the direction normal to the boundary, respectively. Problems of this type occur, e.g., in linear elasticity, where w is the deflection of a clamped Kirchhoff plate under a vertical load with density f.

In this paper we focus on finite element methods for discretizing the continuous problem (1.1). The aim is to construct and analyze efficient iterative methods for solving the resulting linear system. In particular, the Hellan-Herrmann-Johnson (HHJ) mixed finite element method is studied, see [16, 17, 19], which is strongly related to the non-conforming Morley finite element, see [21, 2]. The proposed iterative method consists of the application of the preconditioned conjugate gradient method to three discretized elliptic problems of second order. The implementation requires only manipulations with standard conforming Lagrangian finite elements for second-order problems. The proposed preconditioners are standard multigrid preconditioners for second-order problems, which lead to mesh-independent convergence rates.

The results are based on a decomposition of the continuous problem into three second-order elliptic problems, which are to be solved consecutively. The first and the last problem are Poisson problems with Dirichlet conditions, the second problem is a pure traction problem in planar linear elasticity with Poisson ratio 0. The HHJ method is a non-conforming method in this setup. A conforming modification will be discussed as well.

There are many alternative approaches for biharmonic problems discussed in literature. Finite element discretizations range from conforming and classical non-conforming finite element methods for fourth-order problems, discontinuous Galerkin methods for fourth-order

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problems to various mixed methods, see, e.g., [10, 12, 8, 4], and the references cited there. Solution techniques proposed for the linear systems, which show mesh-independent or nearly mesh-independent convergence rates are typically based on two-level or multilevel additive or multiplicative Schwarz methods (including multigrid methods), see, e.g., [24, 7, 26, 15], and the references cited there.

We are not aware of any other approach, which is based solely on standard components for second-order elliptic problems and for which optimal convergence behavior could be shown. An additional feature of the approach in this paper is a new formulation of the underlying continuous mixed variational problem, which is fully equivalent to the original primal variational problem without any further assumptions on Ω like convexity. This was achieved by introducing an appropriate nonstandard Sobolev space.

The paper is organized as follows. Section 2 contains a modification of a standard mixed formulation of the biharmonic problem, for which well-posedness will be shown. A Helmholtz decomposition of an involved nonstandard Sobolev space is derived in Section 3 and the resulting decomposition of the biharmonic problem is presented. In Section 4 the Hellan-Herrmann-Johnson method is discussed. Section 5 contains the discrete version of the Helmholtz decomposition of Section 3. The paper closes with a few numerical experiments in Section 6 for illustrating the theoretical results.

2. A modified mixed variational formulation. Here and throughout the paper we use $L^2(\Omega)$, $H^m(\Omega)$, and $H_0^m(\Omega)$ with its dual space $H^{-m}(\Omega)$ to denote the standard Lebesgue and Sobolev spaces with corresponding norms $\|.\|_0$, $\|.\|_m$, $|.|_m$, and $\|.\|_{-m}$ for positive integers m, see, e.g., [1].

A standard (primal) variational formulation of (1.1) reads as follows: For given $f \in H^{-1}(\Omega)$, find $w \in H_0^2(\Omega)$ such that

$$\int_\Omega \nabla^2 w : \nabla^2 v \, dx = \langle f, v\rangle \quad \text{for all } v \in H_0^2(\Omega), \tag{2.1}$$

where ∇^2 denotes the Hessian, $\boldsymbol{A} : \boldsymbol{B} = \sum_{i,j=1}^2 \boldsymbol{A}_{ij}\, \boldsymbol{B}_{ij}$ for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{2\times 2}$, and $\langle \cdot, \cdot\rangle$ denotes the duality product in $H^* \times H$ for a Hilbert space H with dual H^*, here for $H = H_0^1(\Omega)$. Existence and uniqueness of a solution to (2.1) is guaranteed even for more general right hand sides $f \in H^{-2}(\Omega)$ by the theorem of Lax-Milgram, see, e.g., [22, 20].

For the HHJ mixed method the auxiliary variable

$$\boldsymbol{w} = \nabla^2 w \tag{2.2}$$

is introduced, whose elements can be interpreted as bending moments in the context of linear elasticity. This allows to rewrite the biharmonic equation in (1.1) as a system of two second-order equations

$$\nabla^2 w = \boldsymbol{w}, \quad \operatorname{div} \operatorname{\mathbf{div}} \boldsymbol{w} = f \quad \text{in } \Omega \tag{2.3}$$

with the following notations for a matrix-valued function $\boldsymbol{v}$ and a vector-valued function ϕ.

$$\operatorname{\mathbf{div}} \boldsymbol{v} = \begin{bmatrix} \partial_1 \boldsymbol{v}_{11} + \partial_2 \boldsymbol{v}_{12} \\ \partial_1 \boldsymbol{v}_{21} + \partial_2 \boldsymbol{v}_{22} \end{bmatrix} \quad \text{and} \quad \operatorname{div} \phi = \partial_1 \phi_1 + \partial_2 \phi_2.$$

In the standard approach a mixed variational formation is directly derived from the system (2.3). We take here a little detour, which better motivates the nonstandard Sobolev space we use in this paper for a modified mixed variational formulation. Starting point is the following unconstrained optimization problem: Find $w \in H_0^2(\Omega)$ such that

$$J(w) = \min_{v \in H_0^2(\Omega)} J(v) \tag{2.4}$$

with

$$J(v) = \frac{1}{2}\int_\Omega \nabla^2 v : \nabla^2 v \, dx - \langle f, v\rangle.$$

It is well-known that (2.4) is equivalent to (2.1). Actually, (2.1) can be seen as the optimality system characterizing the solution of (2.4). By introducing the auxiliary variable $\boldsymbol{w} = \nabla^2 w \in \boldsymbol{L}^2(\Omega)_{\text{sym}}$ with

$$\boldsymbol{L}^2(\Omega)_{\text{sym}} = \{\boldsymbol{v}\colon \boldsymbol{v}_{ji} = \boldsymbol{v}_{ij} \in L^2(\Omega),\ i,j=1,2\},$$

equipped with the standard L^2-norm $\|\boldsymbol{v}\|_0$ for matrix-valued functions $\boldsymbol{v}$, the objective functional becomes a functional depending on the original and the auxiliary variable:

$$J(v,\boldsymbol{v}) = \frac{1}{2}\int_\Omega \boldsymbol{v} : \boldsymbol{v} \, dx - \langle f, v\rangle. \tag{2.5}$$

The weak formulation of (2.2) leads to the constraint

$$c((w,\boldsymbol{w}),\boldsymbol{\tau}) = 0 \quad \text{for all } \boldsymbol{\tau} \in \boldsymbol{M}, \tag{2.6}$$

where

$$c((v,\boldsymbol{v}),\boldsymbol{\tau}) = -\int_\Omega \boldsymbol{v} : \boldsymbol{\tau} \, dx - \int_\Omega \nabla v \cdot \operatorname{\mathbf{div}} \boldsymbol{\tau} \, dx,$$

and $\boldsymbol{M}$ is a (not yet specified) space of sufficiently smooth matrix-valued test functions. By this the unconstrained optimization problem (2.4) is transformed to the following constrained optimization problem: Find $(w,\boldsymbol{w}) \in H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)_{\text{sym}}$ that minimizes the objective functional (2.5) subject to the constraint (2.6). The Lagrangian functional associated with this constrained optimization problem is given by

$$\mathscr{L}((v,\boldsymbol{v}),\boldsymbol{\tau}) = J(v,\boldsymbol{v}) + c((v,\boldsymbol{v}),\boldsymbol{\tau}),$$

which leads to the following first-order optimality system:

$$\begin{aligned} &\int_\Omega \boldsymbol{w} : \boldsymbol{v} \, dx + c((v,\boldsymbol{v}),\boldsymbol{\sigma}) = \langle f, v\rangle && \text{for all } (v,\boldsymbol{v}) \in H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)_{\text{sym}}, \\ &c((w,\boldsymbol{w}),\boldsymbol{\tau}) \qquad\qquad\quad\; = 0 && \text{for all } \boldsymbol{\tau} \in \boldsymbol{M}. \end{aligned} \tag{2.7}$$

Here $\boldsymbol{\sigma} \in \boldsymbol{M}$ denotes the Lagrangian multiplier associated with the constraint (2.6). The optimality system is a saddle point problem on the space $\boldsymbol{X} = H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)_{\text{sym}}$, equipped with the standard norm

$$\|(v,\boldsymbol{v})\|_{\boldsymbol{X}} = \left(|v|_1^2 + \|\boldsymbol{v}\|_0^2\right)^{1/2}$$

for the primal variable $(v,\boldsymbol{v})$ and the (not yet specified) Hilbert space $\boldsymbol{M}$, equipped with a norm $\|\boldsymbol{\tau}\|_{\boldsymbol{M}}$ for the dual variable $\boldsymbol{\tau}$. An essential condition for the analysis of (2.7) is the inf-sup condition for the bilinear form c, which reads: There is a constant $\beta > 0$ such that

$$\sup_{0 \neq (v,\boldsymbol{v}) \in \boldsymbol{X}} \frac{c((v,\boldsymbol{v}),\boldsymbol{\tau})}{\|(v,\boldsymbol{v})\|_{\boldsymbol{X}}} \geq \beta \, \|\boldsymbol{\tau}\|_{\boldsymbol{M}},$$

It is easy to see that

$$\sup_{0 \neq (v,\boldsymbol{v}) \in \boldsymbol{X}} \frac{c((v,\boldsymbol{v}),\boldsymbol{\tau})}{\|(v,\boldsymbol{v})\|_{\boldsymbol{X}}} = \left(\|\boldsymbol{\tau}\|_0^2 + \|\operatorname{div}\operatorname{\mathbf{div}}\boldsymbol{\tau}\|_{-1}^2\right)^{1/2} \tag{2.8}$$

for sufficiently smooth functions $\boldsymbol{\tau}$. If the right-hand side in (2.8) is chosen as the norm in $\boldsymbol{M}$, then the inf-sup condition is trivially satisfied with constant $\beta = 1$. This motivates to set $\boldsymbol{M} = \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ with

$$\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}} = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}} \colon \operatorname{div}\mathbf{div}\,\boldsymbol{\tau} \in H^{-1}(\Omega)\},$$

equipped with the norm

$$\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div}} = \left(\|\boldsymbol{\tau}\|_0^2 + \|\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}\|_{-1}^2\right)^{1/2}. \tag{2.9}$$

Here $\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}$ is meant in the distributional sense. It is easy to see that $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ is a Hilbert space. In order to have a well-defined bilinear form c, the original definition has to be replaced by

$$c((v,\boldsymbol{v}),\boldsymbol{\tau}) = -\int_\Omega \boldsymbol{v} : \boldsymbol{\tau}\;dx + \langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, v\rangle,$$

which coincides with the original definition, if $\boldsymbol{\tau}$ is sufficiently smooth, say $\boldsymbol{\tau} \in \boldsymbol{H}^1(\Omega)_{\text{sym}}$ with

$$\boldsymbol{H}^1(\Omega)_{\text{sym}} = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}} \colon \boldsymbol{\tau}_{ij} \in H^1(\Omega),\ i,j = 1,2\},$$

equipped with the standard H^1-norm $\|\boldsymbol{\tau}\|_1$ and H^1-semi-norm $|\boldsymbol{\tau}|_1$ for matrix-valued functions $\boldsymbol{\tau}$. Observe that

$$\boldsymbol{H}^1(\Omega)_{\text{sym}} \subset \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}} \subset \boldsymbol{L}^2(\Omega)_{\text{sym}}.$$

From the first row of the optimality system (2.7) for $v = 0$ it easily follows that $\boldsymbol{w} = \boldsymbol{\sigma}$. So the auxiliary variable $\boldsymbol{w}$ can be eliminated and we obtain after reordering the following reduced optimality system: For $f \in H^{-1}(\Omega)$, find $\boldsymbol{\sigma} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}}$ and $w \in H_0^1(\Omega)$ such that

$$\begin{aligned} &\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\;dx && - \langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, w\rangle = 0 && \text{for all } \boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}},\\ &- \langle \operatorname{div}\mathbf{div}\,\boldsymbol{\sigma}, v\rangle && = -\langle f, v\rangle && \text{for all } v \in H_0^1(\Omega). \end{aligned} \tag{2.10}$$

REMARK 2.1. The presented approach to derive the mixed method via the optimality system of a constrained optimization problem is the same approach as taken in [11] for the Ciarlet-Raviart mixed method. See [27] for a reformulation involving a similar nonstandard Sobolev space $H^{-1}(\Delta,\Omega) = \{v \in H^1(\Omega) : \Delta v \in H^{-1}(\Omega)\}$ as in this paper.

Problem (2.10) has the typical structure of a saddle point problem

$$\begin{aligned} a(\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\boldsymbol{\tau}, w) &= 0 && \text{for all } \boldsymbol{\tau} \in \boldsymbol{V},\\ b(\boldsymbol{\sigma}, v) &= -\langle f, v\rangle && \text{for all } v \in Q. \end{aligned} \tag{2.11}$$

If the linear operator $\mathcal{A} \colon \boldsymbol{V} \times Q \longrightarrow (\boldsymbol{V} \times Q)^*$ is introduced by

$$\left\langle \mathcal{A}\begin{bmatrix}\boldsymbol{\sigma}\\ w\end{bmatrix}, \begin{bmatrix}\boldsymbol{\tau}\\ v\end{bmatrix}\right\rangle = a(\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\boldsymbol{\tau}, w) + b(\boldsymbol{\sigma}, v),$$

the mixed variational problem (2.11) can be rewritten as a linear operator equation

$$\mathcal{A}\begin{bmatrix}\boldsymbol{\sigma}\\ w\end{bmatrix} = -\begin{bmatrix}0\\ f\end{bmatrix}.$$

If the bilinear form a is symmetric, i.e., $a(\boldsymbol{\sigma},\boldsymbol{\tau}) = a(\boldsymbol{\tau},\boldsymbol{\sigma})$, and non-negative, i.e., $a(\boldsymbol{\tau},\boldsymbol{\tau}) \geq 0$, which is the case for (2.10), it is well-known that $\mathcal{A}$ is an isomorphism from $\boldsymbol{V} \times Q$ onto $(\boldsymbol{V} \times Q)^*$, if and only if the following conditions are satisfied, see, e.g., [6]:

1. a is bounded: There is a constant $\|a\| > 0$ such that

$$|a(\boldsymbol{\sigma}, \boldsymbol{\tau})| \leq \|a\| \, \|\boldsymbol{\sigma}\|_{\boldsymbol{V}} \, \|\boldsymbol{\tau}\|_{\boldsymbol{V}} \quad \text{for all } \boldsymbol{\sigma}, \ \boldsymbol{\tau} \in \boldsymbol{V}.$$

2. b is bounded: There is a constant $\|b\| > 0$ such that

$$|b(\boldsymbol{\tau}, v)| \leq \|b\| \, \|\boldsymbol{\tau}\|_{\boldsymbol{V}} \|v\|_Q \quad \text{for all } \boldsymbol{\tau} \in \boldsymbol{V}, \ v \in Q.$$

3. a is coercive on the kernel of b: There is a constant $\alpha > 0$ such that

$$a(\boldsymbol{\tau}, \boldsymbol{\tau}) \geq \alpha \, \|\boldsymbol{\tau}\|_{\boldsymbol{V}}^2 \quad \text{for all } \boldsymbol{\tau} \in \ker B$$

with $\ker B = \{\boldsymbol{\tau} \in \boldsymbol{V} \colon b(\boldsymbol{\tau}, v) = 0 \text{ for all } v \in Q\}$.
4. b satisfies the inf-sup condition: There is a constant $\beta > 0$ such that

$$\inf_{0 \neq v \in Q} \sup_{0 \neq \boldsymbol{\tau} \in \boldsymbol{V}} \frac{b(\boldsymbol{\tau}, v)}{\|\boldsymbol{\tau}\|_{\boldsymbol{V}} \, \|v\|_Q} \geq \beta.$$

Here $\|\boldsymbol{\tau}\|_{\boldsymbol{V}}$ and $\|v\|_Q$ denote the norms in $\boldsymbol{V}$ and Q, respectively. We will refer to theses conditions as Brezzi's conditions with constants $\|a\|$, $\|b\|$, α, and β. (We silently assume that $\|a\|$ and $\|b\|$ are the smallest constants for estimating the bilinear forms a and b. Then $\|a\|$ and $\|b\|$ match the standard notation for the norms of the bilinear forms a and b.)

In the next theorem we show that Brezzi's conditions are satisfied for (2.10). For the proof as well as for later use, we first introduce the following simple but useful notation for a function $v \in H_0^1(\Omega)$:

$$\boldsymbol{\pi}(v) = v \, \boldsymbol{I} \quad \text{with} \quad \boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.12}$$

THEOREM 2.2. *The bilinear forms*

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} \, dx \quad \textit{and} \quad b(\boldsymbol{\tau}, v) = -\langle \operatorname{div} \mathbf{div} \, \boldsymbol{\tau}, v \rangle$$

satisfy Brezzi's conditions on $\boldsymbol{V} = \boldsymbol{H}^{-1}(\operatorname{div} \mathbf{div}, \Omega)_{sym}$ *and* $Q = H_0^1(\Omega)$*, equipped with the norms* $\|\boldsymbol{\tau}\|_{-1, \operatorname{div} \mathbf{div}}$ *and* $|v|_1$*, respectively, with the constants*

$$\|a\| = \|b\| = \alpha = 1 \quad \textit{and} \quad \beta = (1 + 2c_F^2)^{-1/2},$$

where c_F *denotes the constant in Friedrichs' inequality:* $\|v\|_0 \leq c_F \, |v|_1$ *for all* $v \in H_0^1(\Omega)$.

Proof.

1. Let $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div} \mathbf{div}, \Omega)_{\text{sym}}$. Then

$$|a(\boldsymbol{\sigma}, \boldsymbol{\tau})| \leq \|\boldsymbol{\sigma}\|_0 \|\boldsymbol{\tau}\|_0 \leq \|\boldsymbol{\sigma}\|_{-1, \operatorname{div} \mathbf{div}} \|\boldsymbol{\tau}\|_{-1, \operatorname{div} \mathbf{div}}.$$

2. Let $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div} \mathbf{div}, \Omega)_{\text{sym}}$ and $v \in H_0^1(\Omega)$. Then

$$|b(\boldsymbol{\tau}, v)| = |\langle \operatorname{div} \mathbf{div} \, \boldsymbol{\tau}, v \rangle \leq \| \operatorname{div} \mathbf{div} \, \boldsymbol{\tau}\|_{-1} \, |v|_1 \leq \|\boldsymbol{\tau}\|_{-1, \operatorname{div} \mathbf{div}} \, |v|_1.$$

3. Observe that $\ker B = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}} : \operatorname{div} \mathbf{div} \, \boldsymbol{\tau} = 0\}$. Therefore,

$$a(\boldsymbol{\tau}, \boldsymbol{\tau}) = \|\boldsymbol{\tau}\|_0^2 = \|\boldsymbol{\tau}\|_{-1, \operatorname{div} \mathbf{div}}^2 \quad \text{for all } \boldsymbol{\tau} \in \ker B.$$

4. Here we follow the proofs in [9, 6]. For $v \in H_0^1(\Omega)$ it is easy to see that

$$b(\boldsymbol{\pi}(v), v) = |v|_1^2 \quad \text{and} \quad \|\boldsymbol{\pi}(v)\|^2_{-1,\operatorname{div}\mathbf{div}} = \|\boldsymbol{\pi}(v)\|_0^2 + |v|_1^2 \le (1 + 2c_F^2)\,|v|_1^2.$$

Therefore

$$\begin{aligned}\sup_{0 \neq \boldsymbol{\tau} \in \boldsymbol{V}} \frac{|b(\boldsymbol{\tau}, v)|}{\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div}}} &\ge \frac{|b(\boldsymbol{\pi}(v), v)|}{\|\boldsymbol{\pi}(v)\|_{-1,\operatorname{div}\mathbf{div}}} = \frac{|v|_1^2}{\|\boldsymbol{\pi}(v)\|_0^2 + |v|_1^2)^{1/2}} \\ &\ge \frac{1}{(1+2c_F^2)^{1/2}}\,|v|_1. \qquad \square\end{aligned}$$

COROLLARY 2.3. *The problems (2.1) and (2.10) are fully equivalent, i.e., if $w \in H_0^2(\Omega)$ solves (2.1), then $\boldsymbol{\sigma} = \nabla^2 w \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{sym}$ and $(\boldsymbol{\sigma}, w)$ solves (2.10). And, vice versa, if $(\boldsymbol{\sigma}, w) \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{sym} \times H_0^1(\Omega)$ solves (2.1), then $w \in H_0^2(\Omega)$ and w solves (2.1).*

Proof. Both problems are uniquely solvable. Therefore, it suffices to show that $(w, \boldsymbol{\sigma})$ with $\boldsymbol{\sigma} = \nabla^2 w$ solves (2.10), if w solves (2.1). So, assume that $w \in H_0^2(\Omega)$ is a solution of (2.1). Then, obviously, $\boldsymbol{\sigma} \in \boldsymbol{L}^2(\Omega)_{\text{sym}}$ and

$$\int_\Omega \boldsymbol{\sigma} : \nabla^2 v \; dx = \langle f, v\rangle \quad \text{for all } v \in H_0^2(\Omega),$$

which implies that $\operatorname{div}\mathbf{div}\,\boldsymbol{\sigma} = f \in H^{-1}(\Omega)$ in the distributional sense. Therefore, $\boldsymbol{\sigma} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{\text{sym}}$ and the second row in (2.10) immediately follows.

By the definition of $\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}$ in the distributional sense we have

$$\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, v\rangle = \int_\Omega \boldsymbol{\tau} : \nabla^2 v \,\mathrm{d}x \quad \forall v \in C_0^\infty(\Omega).$$

Since $C_0^\infty(\Omega)$ is dense in $H_0^2(\Omega)$, it follows for $v = w$ that

$$\langle \operatorname{div}\mathbf{div}\,\boldsymbol{\tau}, w\rangle = \int_\Omega \boldsymbol{\tau} : \nabla^2 w \; dx = \int_\Omega \boldsymbol{\tau} : \boldsymbol{\sigma} \; dx,$$

which shows the first row in (2.10). $\square$

REMARK 2.4. The space $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{\text{sym}}$ was already introduced [25, 23] in the context of linear elasticity problems.

There is a natural trace operator associated with $\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{\text{sym}}$, which was discussed in [25, 23]. We shortly recall here the basic properties for later reference.

Let the boundary Γ of the polygonal domain Ω be written in the form

$$\Gamma = \bigcup_{k=1}^{K} \overline{\Gamma}_k,$$

where Γ_k, $k = 1, 2, \ldots, K$, are the edges of Γ, considered as open line segments. $\overline{\Gamma}_k$ denotes the corresponding closed line segment. For $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{\text{sym}}$ which are additionally twice continuously differentiable and $v \in H^2(\Omega) \cap H_0^1(\Omega)$ we obtain the following identity by integration by parts.

$$\int_\Omega (\operatorname{div}\mathbf{div}\,\boldsymbol{\tau})\, v \; dx = \int_\Omega \boldsymbol{\tau} : \nabla^2 v \; dx - \int_\Gamma \boldsymbol{\tau}_{nn} \frac{\partial v}{\partial n} \; ds \tag{2.13}$$

with the outer normal unit vector n of Γ and

$$\boldsymbol{\tau}_{nn} = n^T \boldsymbol{\tau} n.$$

Following standard procedures this identity allows to extend the trace $\boldsymbol{\tau}_{nn}$ to all functions $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{\mathbf{div}}, \Omega)_{\text{sym}}$ as an element of the dual of the image of the Neumann traces of functions from $H^2(\Omega) \cap H_0^1(\Omega)$, i.e.

$$\boldsymbol{\tau}_{nn} \in H_{pw}^{-1/2}(\Gamma) = \Pi_{k=1}^{K} H^{-1/2}(\Gamma_k),$$

where $H^{-1/2}(\Gamma_k)$ is the dual of $\widetilde{H}^{1/2}(\Gamma_k)$, see [14] for details. Another widely used notation for $\widetilde{H}^{1/2}(\Gamma_k)$ is $H_{00}^{1/2}(\Gamma_k)$, see [20].

From (2.13) we obtain the corresponding Green's formula for $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{\mathbf{div}}, \Omega)_{\text{sym}}$ and $v \in H^2(\Omega) \cap H_0^1(\Omega)$:

$$\langle \operatorname{div}\operatorname{\mathbf{div}} \boldsymbol{\tau}, v\rangle = \int_\Omega \boldsymbol{\tau} : \nabla^2 v \, dx - \left\langle \boldsymbol{\tau}_{nn}, \frac{\partial v}{\partial n} \right\rangle_\Gamma . \tag{2.14}$$

Here $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality product in a Hilbert space of functions on Γ.

3. A Helmholtz decomposition of $\boldsymbol{H}^{-1}(\operatorname{div}\operatorname{\mathbf{div}}, \Omega)_{\text{sym}}$. In this section we study some important structural properties of $\boldsymbol{H}^{-1}(\operatorname{div}\operatorname{\mathbf{div}}, \Omega)_{\text{sym}}$, which are helpful for analyzing the HHJ method in the next sections.

THEOREM 3.1. *For each $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{\mathbf{div}}, \Omega)_{\text{sym}}$, there is a unique decomposition*

$$\boldsymbol{\tau} = \boldsymbol{\tau}_0 + \boldsymbol{\tau}_1,$$

where $\boldsymbol{\tau}_0 = \boldsymbol{\pi}(p)$ for some $p \in H_0^1(\Omega)$ and $\boldsymbol{\tau}_1 \in \boldsymbol{L}^2(\Omega)_{\text{sym}}$ with $\operatorname{div}\operatorname{\mathbf{div}} \boldsymbol{\tau}_1 = 0$. Moreover,

$$\underline{c} \, \left(|\boldsymbol{\tau}_0|_1^2 + \|\boldsymbol{\tau}_1\|_0^2 \right) \leq \|\boldsymbol{\tau}\|_{-1,\operatorname{div}\operatorname{\mathbf{div}}}^2 \leq \overline{c} \, \left(|\boldsymbol{\tau}_0|_1^2 + \|\boldsymbol{\tau}_1\|_0^2 \right)$$

for all $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{\mathbf{div}}, \Omega)_{\text{sym}}$, with positive constants $\underline{c}$ and $\overline{c}$ which depend only on the constant c_F of Friedrichs' inequality.

Proof. For $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{\mathbf{div}}, \Omega)_{\text{sym}}$, let $p \in H_0^1(\Omega)$ be the unique solution to the variational problem

$$\int_\Omega \nabla p \cdot \nabla v \, dx = -\langle \operatorname{div}\operatorname{\mathbf{div}} \boldsymbol{\tau}, v\rangle \quad \text{for all } v \in H_0^1(\Omega) \tag{3.1}$$

and set $\boldsymbol{\tau}_0 = \boldsymbol{\pi}(p)$. Since

$$-\langle \operatorname{div}\operatorname{\mathbf{div}} \boldsymbol{\tau}_0, v\rangle = \int_\Omega \nabla p \cdot \nabla v \, dx,$$

it follows that $\operatorname{div}\operatorname{\mathbf{div}} \boldsymbol{\tau}_0 = \operatorname{div}\operatorname{\mathbf{div}} \boldsymbol{\tau}$, and, therefore, $\operatorname{div}\operatorname{\mathbf{div}} \boldsymbol{\tau}_1 = 0$ for $\boldsymbol{\tau}_1 = \boldsymbol{\tau} - \boldsymbol{\tau}_0$ in the distributional sense. On the other hand, if $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + \boldsymbol{\tau}_1$ with $\boldsymbol{\tau}_0 = \boldsymbol{\pi}(p)$ and $\operatorname{div}\operatorname{\mathbf{div}} \boldsymbol{\tau}_1 = 0$, then $-\operatorname{div}\operatorname{\mathbf{div}} \boldsymbol{\tau}_0 = -\operatorname{div}\operatorname{\mathbf{div}} \boldsymbol{\tau} + \operatorname{div}\operatorname{\mathbf{div}} \boldsymbol{\tau}_1 = -\operatorname{div}\operatorname{\mathbf{div}} \boldsymbol{\tau}$, which implies (3.1). This shows the uniqueness.

Furthermore, (3.1) implies $|\boldsymbol{\tau}_0|_1^2 = 2\,|p|_1^2 = 2\,\|\operatorname{div}\operatorname{\mathbf{div}} \boldsymbol{\tau}\|_{-1}^2$. Hence

$$\begin{aligned}
\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\operatorname{\mathbf{div}}}^2 &= \|\boldsymbol{\tau}\|_0^2 + \|\operatorname{div}\operatorname{\mathbf{div}} \boldsymbol{\tau}\|_{-1}^2 = \|\boldsymbol{\tau}_0 + \boldsymbol{\tau}_1\|_0^2 + \frac{1}{2}\,|\boldsymbol{\tau}_0|_1^2 \\
&\leq 2\,\|\boldsymbol{\tau}_0\|_0^2 + 2\,\|\boldsymbol{\tau}_1\|_0^2 + \frac{1}{2}\,|\boldsymbol{\tau}_0|_1^2 \leq \left(\frac{1}{2} + 2c_F^2\right)\,|\boldsymbol{\tau}_0|_1^2 + 2\,\|\boldsymbol{\tau}_1\|_0^2
\end{aligned}$$

and

$$\begin{aligned}|\boldsymbol{\tau}_0|_1^2 + \|\boldsymbol{\tau}_1\|_0^2 &= |\boldsymbol{\tau}_0|_1^2 + \|\boldsymbol{\tau} - \boldsymbol{\tau}_0\|_0^2 \le |\boldsymbol{\tau}_0|_1^2 + 2\,\|\boldsymbol{\tau}\|_0^2 + 2\,\|\boldsymbol{\tau}_0\|_0^2 \\ &\le 2\,\|\boldsymbol{\tau}\|_0^2 + (1+2c_F^2)\,|\boldsymbol{\tau}_0|_1^2 = 2\,\|\boldsymbol{\tau}\|_0^2 + 2(1+2c_F^2)\,\|\operatorname{div}\mathbf{div}\,\boldsymbol{\tau}\|_{-1}^2.\end{aligned}$$

Then the estimates immediately follow with the constants $1/\underline{c} = 2(1+2c_F^2)$ and $\overline{c} = \max(2, 1/2 + 2c_F^2)$. ∎

In short, we have algebraically as well as topologically

$$\boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega)_{\text{sym}} = \boldsymbol{\pi}(H_0^1(\Omega)) \oplus \mathscr{H}(\operatorname{div}\mathbf{div},\Omega)$$

with

$$\mathscr{H}(\operatorname{div}\mathbf{div},\Omega) = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}} \colon\ \operatorname{div}\mathbf{div}\,\boldsymbol{\tau} = 0\}.$$

Here $\oplus$ denotes the direct sum of Hilbert spaces.

REMARK 3.2. The Helmholtz decomposition of $\boldsymbol{L}^2(\Omega)_{\text{sym}}$ in [18], based on previous results in [5], has the same second component. The first component in [5, 18] is different and requires the solution of a biharmonic problem in contrast to Theorem 3.1, where the first component requires to solve only a Poisson problem.

Next an explicit characterization of $\mathscr{H}(\operatorname{div}\mathbf{div},\Omega)$ is given.

THEOREM 3.3. *For each $\boldsymbol{\tau} \in \mathscr{H}(\operatorname{div}\mathbf{div},\Omega)$, there is a function $\phi \in \left(H^1(\Omega)\right)^2$ such that*

$$\boldsymbol{\tau} = \boldsymbol{H}^T\boldsymbol{\varepsilon}(\phi)\boldsymbol{H} \quad \textit{with} \quad \boldsymbol{H} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \quad \textit{and} \quad \boldsymbol{\varepsilon}(\phi)_{ij} = \frac{1}{2}\left(\partial_j\phi_i + \partial_i\phi_j\right).$$

And vice versa, each function of the form $\boldsymbol{\tau} = \boldsymbol{H}^T\boldsymbol{\varepsilon}(\phi)\boldsymbol{H}$ with $\phi \in \left(H^1(\Omega)\right)^2$ lies in $\mathscr{H}(\operatorname{div}\mathbf{div},\Omega)$.

The function ϕ is unique up to an element from

$$RM = \left\{\boldsymbol{\tau}(x) = a\begin{bmatrix}-x_2\\ x_1\end{bmatrix} + b \colon a \in \mathbb{R},\ b \in \mathbb{R}^2\right\},$$

and there is a constant c_K such that

$$c_K\,\|\phi\|_1 \le \|\boldsymbol{\tau}\|_0 = \|\boldsymbol{\varepsilon}(\phi)\|_0 \le \|\phi\|_1 \quad \textit{for all } \phi \in \left(H^1(\Omega)\right)^2/RM.$$

Proof. In [18] it was shown that $\boldsymbol{\tau} \in \mathscr{H}(\operatorname{div}\mathbf{div},\Omega)$ can be written in the following way:

$$\boldsymbol{\tau} = \begin{bmatrix}0 & -\rho\\ \rho & 0\end{bmatrix} + \mathbf{Curl}\,\phi \quad \text{with} \quad \rho = \frac{1}{2}\operatorname{div}\phi, \quad \mathbf{Curl}\,\phi = \begin{bmatrix}-\partial_2\phi_1 & \partial_1\phi_1\\ -\partial_2\phi_2 & \partial_1\phi_2\end{bmatrix}$$

for some $\phi \in \left(H^1(\Omega)\right)^2$. Replacing $\phi = (\phi_1,\phi_2)^T$ by $(-\phi_2,\phi_1)^T$ yields the representation. The estimates follow from Korn's inequality. ∎

Therefore, we have the following representation of the solution $\boldsymbol{\sigma}$ to (2.10):

$$\boldsymbol{\sigma} = \boldsymbol{\pi}(p) + \boldsymbol{H}^T\boldsymbol{\varepsilon}(\phi)\boldsymbol{H}.$$

The analogous representation for the test functions $\boldsymbol{\tau} = \boldsymbol{\pi}(q) + \boldsymbol{H}^T \boldsymbol{\varepsilon}(\psi) \boldsymbol{H}$ leads to following equivalent formulation of (2.10). Find $p \in H_0^1(\Omega)$, $\phi \in \left(H^1(\Omega)\right)^2/\mathrm{RM}$, $w \in H_0^1(\Omega)$ such that

$$
\begin{aligned}
&\int_\Omega \boldsymbol{\pi}(p) : \boldsymbol{\pi}(q)\,dx + \int_\Omega \boldsymbol{\pi}(q) : \boldsymbol{\varepsilon}(\phi)\,dx + \int_\Omega \nabla w \cdot \nabla q\,dx = 0,\\
&\int_\Omega \boldsymbol{\pi}(p) : \boldsymbol{\varepsilon}(\psi)\,dx + \int_\Omega \boldsymbol{\varepsilon}(\phi) : \boldsymbol{\varepsilon}(\psi)\,dx \qquad\qquad\qquad = 0\\
&\int_\Omega \nabla p \cdot \nabla v\,dx \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = -\langle f, v\rangle
\end{aligned}
\tag{3.2}
$$

for all $q \in H_0^1(\Omega)$, $\psi \in \left(H^1(\Omega)\right)^2/\mathrm{RM}$, $v \in H_0^1(\Omega)$.

Observe that $\boldsymbol{\pi}(p) : \boldsymbol{\pi}(q) = 2\,p\,q$ and $\boldsymbol{\pi}(q) : \boldsymbol{\varepsilon}(\psi) = q\ \mathrm{div}\,\psi$, which allows to simplify parts of (3.2).

In summary, the biharmonic problem is equivalent to three (consecutively to solve) elliptic second-order problems. The first problem is a Poisson problem with Dirichlet boundary conditions for p, which reads in strong form

$$\Delta p = f \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma.$$

The second problem is a pure traction problem in linear elasticity with Poisson ratio 0 for ϕ, which reads in strong form

$$-\,\mathbf{div}\,\boldsymbol{\varepsilon}(\phi) = \nabla p \quad \text{in } \Omega, \quad \boldsymbol{\varepsilon}(\phi)\,n = 0 \quad \text{on } \Gamma.$$

And, finally, the third problem is a Poisson problem with Dirichlet boundary conditions for the original variable w, which reads in strong form

$$\Delta w = 2\,p + \mathrm{div}\,\phi \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \Gamma.$$

4. The Hellan-Herrmann-Johnson method. Let $\mathcal{T}_h$ be an admissible triangulation of the polygonal domain Ω. For $k \in \mathbb{N}$ the standard finite element spaces $\mathcal{S}_h$ and $\mathcal{S}_{h,0}$ are given by

$$\mathcal{S}_h = \{v \in C(\overline{\Omega})\colon v|_T \in P_k \text{ for all } T \in \mathcal{T}_h\} \quad \text{and} \quad \mathcal{S}_{h,0} = \mathcal{S}_h \cap H_0^1(\Omega),$$

where P_k denotes the set of bivariate polynomials of total degree less than or equal to k.

For the approximation of the Lagrangian multiplier $\boldsymbol{\sigma}$, the HHJ method uses the finite element space

$$
\begin{aligned}
\boldsymbol{V}_h = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}}\colon \boldsymbol{\tau}|_T \in P_{k-1} \text{ for all } T \in \mathcal{T}_h, \text{ and}\\
\boldsymbol{\tau}_{nn} \text{ is continuous across inter-element boundaries}\}.
\end{aligned}
$$

For the approximation of the original variable w the standard finite element space

$$Q_h = \mathcal{S}_{h,0}$$

is used. So, the HHJ method reads as follows: Find $\boldsymbol{\sigma}_h \in \boldsymbol{V}_h$ and $w_h \in Q_h$ such that

$$
\begin{aligned}
&\int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\tau}\,dx \quad - \langle \mathrm{div}\,\mathbf{div}_h\,\boldsymbol{\tau}, w_h\rangle = 0 && \text{for all } \boldsymbol{\tau} \in \boldsymbol{V}_h,\\
&-\langle \mathrm{div}\,\mathbf{div}_h\,\boldsymbol{\sigma}_h, v\rangle \qquad\qquad\quad = -\langle f, v\rangle && \text{for all } v \in Q_h
\end{aligned}
\tag{4.1}
$$

with

$$\langle \operatorname{div} \mathbf{div}_h \boldsymbol{\tau}, v\rangle = \sum_T \left\{ \int_T \boldsymbol{\tau} : \nabla^2 v \, dx - \int_{\partial T} \boldsymbol{\tau}_{nn} \frac{\partial v}{\partial n} \, ds \right\} \quad \text{for } \boldsymbol{\tau} \in \boldsymbol{V}_h, \ v \in Q_h.$$

If compared with (2.14), this definition of $\langle \operatorname{div} \mathbf{div}_h \boldsymbol{\tau}, v\rangle$ for $\boldsymbol{\tau} \in \boldsymbol{V}_h$ and $v \in Q_h$ in the HHJ method is just an element-wise assembled version of corresponding expression on the continuous level, a standard technique in non-conforming methods.

REMARK 4.1. Using integration by parts we obtain

$$\langle \operatorname{div} \mathbf{div}_h \boldsymbol{\tau}, v\rangle = -\sum_{T \in \mathcal{T}_h} \left\{ \int_T \mathbf{div}\, \boldsymbol{\tau} \cdot \nabla v \, dx - \int_{\partial T} \boldsymbol{\tau}_{ns} \frac{\partial v}{\partial s} \, ds \right\}$$

with the normal vector $n = (n_1, n_2)^T$, the vector $s = (-n_2, n_1)^T$, which is tangent to Γ, the tangential derivative $\partial v / \partial s$, and

$$\boldsymbol{\tau}_{ns} = s^T \boldsymbol{\tau} n.$$

The HHJ method is often formulated with this representation, which allows an extension for all functions $\boldsymbol{\tau}$ from the (mesh-dependent) infinite dimensional function space

$$\widetilde{\boldsymbol{V}} = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)_{\text{sym}} \colon \boldsymbol{\tau}|_T \in \boldsymbol{H}^1(T)_{\text{sym}} \text{ for all } T \in \mathcal{T}_h, \text{ and } \\ \boldsymbol{\tau}_{nn} \text{ is continuous across inter-element boundaries}\}.$$

This space was used for the analysis of the method in [9, 3, 13], and others. Existence and uniqueness of a solution for the corresponding variational problem on the continuous level could be shown under additional smoothness assumptions. For the approach taken in this paper, this is not required.

Similar to the continuous case, the well-posedness of the discrete problem can be shown. For the proof of the discrete inf-sup condition the discrete analogue to $\boldsymbol{\pi}(v)$, see (2.12), is needed. For $v_h \in \mathcal{S}_{h,0}$, we define

$$\boldsymbol{\pi}_h(v_h) = \boldsymbol{\Pi}_h \boldsymbol{\pi}(v_h)$$

with the linear operator $\boldsymbol{\Pi}_h$, introduced in [9] by the conditions

$$\int_e ((\boldsymbol{\tau}_h)_{nn} - \boldsymbol{\tau}_{nn}) \, q \, ds = 0, \quad \text{for all } q \in P_{k-1}, \text{ for all edges } e \text{ of } T, \ T \in \mathcal{T}_h,$$

and

$$\int_T (\boldsymbol{\tau}_h - \boldsymbol{\tau}) \, q \, dx = 0, \quad \text{for all } q \in P_{k-2}, \ T \in \mathcal{T}_h,$$

for $\boldsymbol{\tau}_h = \boldsymbol{\Pi}_h \boldsymbol{\tau} \in \boldsymbol{V}_h$ and $\boldsymbol{\tau} \in \boldsymbol{\pi}(Q_h)$. Observe that $\boldsymbol{\Pi}_h$ was originally introduced in [9] as a linear operator on the infinite dimensional space $\widetilde{\boldsymbol{V}}$ from above.

From the corresponding properties of $\boldsymbol{\Pi}_h$ in [9], Lemma 4, the next result directly follows.

LEMMA 4.2. *Assume that $\mathcal{T}_h$ is a regular family of triangulation. Then there exists a constant $c_B > 0$ which is independent of h such that*

$$\|\boldsymbol{\pi}_h(v)\|_0 \le c_B \, |v|_1 \quad \textit{for all } v \in \mathcal{S}_{h,0}.$$

Moreover, we need the following simple identity.

LEMMA 4.3. *For all $p, v \in \mathcal{S}_{h,0}$, we have*

$$-\langle \operatorname{div} \mathbf{div}_h \boldsymbol{\pi}_h(p), v\rangle = \int_\Omega \nabla p \cdot \nabla v \, dx.$$

Proof. By integration by parts we have

$$\begin{aligned}
\langle \operatorname{div} \mathbf{div}_h \boldsymbol{\pi}_h(p), v\rangle &= \sum_{T\in\mathcal{T}_h} \left\{ \int_T \mathbf{\Pi}_h \boldsymbol{\pi}(p) : \nabla^2 v \, dx - \int_{\partial T} (\mathbf{\Pi}_h \boldsymbol{\pi}(p))_{nn} \frac{\partial v}{\partial n} \, ds \right\} \\
&= \sum_{T\in\mathcal{T}_h} \left\{ \int_T \boldsymbol{\pi}(p) : \nabla^2 v \, dx - \int_{\partial T} (\boldsymbol{\pi}(p))_{nn} \frac{\partial v}{\partial n} \, ds \right\} \\
&= \sum_{T\in\mathcal{T}_h} \left\{ \int_T p \, \Delta v \, dx - \int_{\partial T} p \frac{\partial v}{\partial n} \, ds \right\} = -\int_\Omega \nabla p \cdot \nabla v \, dx. \qquad \square
\end{aligned}$$

Now the well-posedness of the discrete problem can be shown.

THEOREM 4.4. *The bilinear forms*

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} \, dx, \quad b_h(\boldsymbol{\tau}, v) = -\langle \operatorname{div} \mathbf{div}_h \boldsymbol{\tau}, v\rangle$$

satisfy Brezzi's conditions on $\boldsymbol{V}_h$ and Q_h, equipped with the norms $\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div},h}$ and $|v|_1$, respectively, where

$$\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div},h} = \left(\|\boldsymbol{\tau}\|_0^2 + \|\operatorname{div} \mathbf{div}_h \boldsymbol{\tau}\|_{-1,h}^2 \right)^{1/2} \tag{4.2}$$

and

$$\|\ell\|_{-1,h} = \sup_{v_h \in S_{h,0}} \frac{|\langle \ell, v_h\rangle|}{|v_h|_1} \quad \textit{for } \ell \in (\mathcal{S}_{h,0})^*,$$

with the constants

$$\|a\| = \|b\| = \alpha = 1 \quad \textit{and} \quad \beta = (1 + c_B^2)^{-1/2},$$

where c_B denotes the constant in Lemma 4.2.

Proof.

1. Let $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{V}_h$. Then

$$|a(\boldsymbol{\sigma}, \boldsymbol{\tau})| \le \|\boldsymbol{\sigma}\|_0 \, \|\boldsymbol{\tau}\|_0 \le \|\boldsymbol{\sigma}\|_{-1,\operatorname{div}\mathbf{div},h} \, \|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div},h}.$$

2. Let $\boldsymbol{\tau} \in \boldsymbol{V}_h$ and $v \in Q_h$. Then

$$|b(\boldsymbol{\tau}, v)| = |\langle \operatorname{div} \mathbf{div}_h \boldsymbol{\tau}, v\rangle \le \|\operatorname{div} \mathbf{div}_h \boldsymbol{\tau}\|_{-1,h} |v|_1 \le \|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div},h} \|v\|_1.$$

3. Observe that $\ker B_h = \{\boldsymbol{\tau} \in \boldsymbol{V}_h : \operatorname{div} \mathbf{div}\, \boldsymbol{\tau}_h = 0\}$. Therefore,

$$a(\boldsymbol{\tau}, \boldsymbol{\tau}) = \|\boldsymbol{\tau}\|_0^2 = \|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div},h}^2 \quad \text{for } \boldsymbol{\tau} \in \ker B_h.$$

4. From Lemma 4.2 and Lemma 4.3 we obtain for $v \in Q_h$

$$b_h(\boldsymbol{\pi}_h(v), v) = |v|_1^2$$

and

$$\|\boldsymbol{\pi}_h(v)\|_{-1,\operatorname{div}\mathbf{div},h}^2 = \|\boldsymbol{\pi}_h(v)\|_0^2 + |v|_1^2 \leq (1 + c_B^2)\,|v|_1^2.$$

Therefore,

$$\sup_{0 \neq \boldsymbol{\tau} \in \boldsymbol{V}_h} \frac{|b_h(\boldsymbol{\tau}, v)|}{\|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div},h}} \geq \frac{|b_h(\boldsymbol{\pi}_h(v), v)|}{\|\boldsymbol{\pi}_h(v)\|_{-1,\operatorname{div}\mathbf{div},h}} = \frac{|v|_1^2}{(\|\boldsymbol{\pi}_h(v)\|_0^2 + |v|_1^2)^{1/2}}$$
$$\geq \frac{1}{(1 + c_B^2)^{1/2}} |v|_1. \qquad \square$$

Observe that the norms introduced for the space $\boldsymbol{V} = \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div}, \Omega)_{\mathrm{sym}}$ in (2.9) and its discrete counterpart $\boldsymbol{V}_h$ in (4.2) are similar but different. For the discrete problem the norm is mesh-dependent.

5. A discrete Helmholtz decomposition. We have the following discrete version of Theorem 3.1.

THEOREM 5.1. *For each $\boldsymbol{\tau} \in \boldsymbol{V}_h$, there is a unique decomposition*

$$\boldsymbol{\tau} = \hat{\boldsymbol{\tau}}_0 + \hat{\boldsymbol{\tau}}_1,$$

where $\hat{\boldsymbol{\tau}}_0 = \boldsymbol{\pi}_h(\hat{p})$ for some $\hat{p} \in Q_h$ and $\hat{\boldsymbol{\tau}}_1 \in \boldsymbol{V}_h$ with $\operatorname{div}\mathbf{div}_h \hat{\boldsymbol{\tau}}_1 = 0$. Moreover,

$$\underline{c}\ \left(|\hat{\boldsymbol{\tau}}_0|_1^2 + \|\hat{\boldsymbol{\tau}}_1\|_0^2\right) \leq \|\boldsymbol{\tau}\|_{-1,\operatorname{div}\mathbf{div},h}^2 \leq \overline{c}\ \left(|\hat{\boldsymbol{\tau}}_0|_1^2 + \|\hat{\boldsymbol{\tau}}_1\|_0^2\right)$$

for all $\boldsymbol{\tau} \in \boldsymbol{V}_h$, with positive constants $\underline{c}$ and $\overline{c}$, which depend only on the constant c_B of the inequality in Lemma 4.2.

The proof is completely analogous to the proof for the continuous case and is, therefore, omitted. The only difference is the use of the estimate from Lemma 4.2 instead of Friedrichs' inequality.

So, in short,

$$\boldsymbol{V}_h = \boldsymbol{\pi}_h(\mathcal{S}_{h,0}) \oplus \mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega)$$

with

$$\mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega) = \{\boldsymbol{v}_h \in \boldsymbol{V}_h \colon \langle \operatorname{div}\mathbf{div}_h \boldsymbol{v}_h, v_h\rangle = 0 \text{ for all } v_h \in Q_h\}.$$

For describing the space $\mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega)$ more explicitly, we consider the subspace of all functions in $\mathscr{H}(\operatorname{div}\mathbf{div}, \Omega)$ which can be represented by a finite element function $\phi \in (\mathcal{S}_h)^2$, for which we show the following result.

THEOREM 5.2. $\mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega) = \{\boldsymbol{\tau} = \boldsymbol{H}^T \boldsymbol{\varepsilon}(\phi) \boldsymbol{H} \colon \phi \in (\mathcal{S}_h)^2\}$.

Proof. Let be $\phi \in (\mathcal{S}_h)^2$. Then $\boldsymbol{\tau} = \boldsymbol{H}^T \boldsymbol{\varepsilon}(\phi) \boldsymbol{H} \in P_{k-1}$ for all triangles $T \in \mathcal{T}_h$. Furthermore, let e be an edge of a triangle T with outer unit normal vector $n = (n_1, n_2)^T$ and unit tangent vector $s = (-n_2, n_1)^T$. By elementary computations we obtain

$$\boldsymbol{\tau}_{nn} = n^T \boldsymbol{H}^T \boldsymbol{\varepsilon}(\phi) \boldsymbol{H} n = s \cdot \frac{\partial \phi}{\partial s}.$$

So, $\boldsymbol{\tau}_{nn}$ depends only on values of ϕ on the edge e, which immediately implies that $\boldsymbol{\tau}_{nn}$ is continuous on inter-element boundaries. This shows that $\boldsymbol{\tau}$ lies in $\boldsymbol{V}_h$, and, therefore, the inclusion $\{\boldsymbol{\tau} = \boldsymbol{H}^T\boldsymbol{\varepsilon}(\phi)\boldsymbol{H} \colon \phi \in (\mathcal{S}_h)^2\} \subset \mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega)$ follows.

The equality follows by comparing the dimensions. We have

$$\dim\{\boldsymbol{\tau} = \boldsymbol{H}^T\boldsymbol{\varepsilon}(\phi)\boldsymbol{H} \colon \phi \in (\mathcal{S}_h)^2\} = 2\dim\mathcal{S}_h - \dim\mathrm{RM} = 2\dim\mathcal{S}_h - 3.$$

On the other hand, by Theorem 5.1, it follows that

$$\dim\mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega) = \dim\boldsymbol{V}_h - \dim\mathcal{S}_{h,0}.$$

A simple count of the degrees of freedom for $\boldsymbol{V}_h$ yields

$$\dim\boldsymbol{V}_h = \dim\mathcal{S}_{h,0} + 2\,\dim\mathcal{S}_h - 3.$$

Therefore, $\mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega) = 2\,\dim\mathcal{S}_h - 3$, which completes the proof. □

REMARK 5.3. A consequence of the last theorem is the important inclusion

$$\mathscr{H}_h(\operatorname{div}\mathbf{div}_h, \Omega) \subset \mathscr{H}(\operatorname{div}\mathbf{div}, \Omega),$$

which resembles the corresponding result of Lemma 5 in [9].

Therefore, we have the following representation of the approximate solution $\boldsymbol{\sigma}_h \in \boldsymbol{V}_h$ of (4.1):

$$\boldsymbol{\sigma}_h = \boldsymbol{\pi}_h(p_h) + \boldsymbol{H}^T\boldsymbol{\varepsilon}_{\boldsymbol{h}}(\phi_h)\boldsymbol{H}.$$

The analogous representation for the test functions $\boldsymbol{\tau} = \boldsymbol{\pi}_h(q) + \boldsymbol{H}^T\boldsymbol{\varepsilon}(\psi)\boldsymbol{H}$ leads to the following equivalent formulation of (4.1). Find $p_h \in \mathcal{S}_{h,0}$, $\phi_h \in (\mathcal{S}_h)^2/\mathrm{RM}$, $w_h \in \mathcal{S}_{h,0}$ such that

$$\begin{aligned}
&\int_\Omega \hat{\boldsymbol{\pi}}_h(p_h) : \hat{\boldsymbol{\pi}}_h(q)dx + \int_\Omega \hat{\boldsymbol{\pi}}_h(q) : \boldsymbol{\varepsilon}(\phi_h)\,dx + \int_\Omega \nabla w_h\cdot\nabla q\,dx = 0,\\
&\int_\Omega \hat{\boldsymbol{\pi}}_h(p_h) : \boldsymbol{\varepsilon}(\psi)\,dx + \int_\Omega \boldsymbol{\varepsilon}(\phi_h) : \boldsymbol{\varepsilon}(\psi)\,dx = 0\\
&\int_\Omega \nabla p_h\cdot\nabla v\,dx = -\langle f, v\rangle
\end{aligned} \tag{5.1}$$

for all $q \in \mathcal{S}_{h,0}$, $\psi \in (\mathcal{S}_h)^2/\mathrm{RM}$, $v \in \mathcal{S}_{h,0}$, and with

$$\hat{\boldsymbol{\pi}}_h(q) = \boldsymbol{H}\boldsymbol{\pi}_h(q)\boldsymbol{H}^T.$$

Observe that the HHJ method, in the form of (5.1) is a non-conforming method for (3.2). An natural modification of the HHJ method is the following conforming variant. Find $p_h \in \mathcal{S}_{h,0}$, $\phi_h \in (\mathcal{S}_h)^2/\mathrm{RM}$, $w_h \in \mathcal{S}_{h,0}$ such that

$$\begin{aligned}
&\int_\Omega \boldsymbol{\pi}(p_h) : \boldsymbol{\pi}(q)dx + \int_\Omega \boldsymbol{\pi}(q) : \boldsymbol{\varepsilon}(\phi_h)\,dx + \int_\Omega \nabla w_h\cdot\nabla q\,dx = 0,\\
&\int_\Omega \boldsymbol{\pi}(p_h) : \boldsymbol{\varepsilon}(\psi)\,dx + \int_\Omega \boldsymbol{\varepsilon}(\phi_h) : \boldsymbol{\varepsilon}(\psi)\,dx = 0\\
&\int_\Omega \nabla p_h\cdot\nabla v\,dx = -\langle f, v\rangle
\end{aligned} \tag{5.2}$$

for all $q \in \mathcal{S}_{h,0}$, $\psi \in (\mathcal{S}_h)^2/\mathrm{RM}$, $v \in \mathcal{S}_{h,0}$. Compared to the non-conforming method, the conforming variant is slightly less costly, since the linear operator $\mathbf{\Pi}_h$ is not needed.

6. Numerical experiments. The obvious procedure for solving (5.1) consists of three consecutive steps.

step 1. For given $f \in H^{-1}(\Omega)$, solve

$$\int_\Omega \nabla p_h \cdot \nabla v \, dx = -\langle f, v \rangle$$

by the preconditioned conjugate gradient (PCG) method with a standard multigrid preconditioner for a Poisson problem.

step 2. For p_h, computed in step 1, solve

$$\int_\Omega \boldsymbol{\varepsilon}(\phi_h) : \boldsymbol{\varepsilon}(\psi) \, dx = -\int_\Omega \hat{\boldsymbol{\pi}}_h(p_h) : \boldsymbol{\varepsilon}(\psi) \, dx \tag{6.1}$$

by the PCG method with a standard multigrid preconditioner for a pure traction problem.

step 3. For p_h and ϕ_h, computed in step 1 and 2, respectively, solve

$$\int_\Omega \nabla w_h \cdot \nabla q \, dx = -\int_\Omega \hat{\boldsymbol{\pi}}_h(p_h) : \hat{\boldsymbol{\pi}}_h(q) \, dx + \int_\Omega \hat{\boldsymbol{\pi}}_h(q) : \boldsymbol{\varepsilon}(\phi_h) \, dx \tag{6.2}$$

by the PCG method with a standard multigrid preconditioner for a Poisson problem.

For the conforming variant (5.2), the right-hand sides in (6.1) and (6.2) have to replaced by the simpler expressions

$$-\int_\Omega \boldsymbol{\pi}(p_h) : \boldsymbol{\varepsilon}(\psi) \, dx = -\int_\Omega p_h \operatorname{div} \psi \, dx$$

and

$$-\int_\Omega \boldsymbol{\pi}(p_h) : \boldsymbol{\pi}(q) dx + \int_\Omega \boldsymbol{\pi}(q) : \boldsymbol{\varepsilon}(\phi_h) \, dx = -2 \int_\Omega p_h \, q \, dx + \int_\Omega q \operatorname{div} \phi_h \, dx,$$

respectively.

For illustrating the theoretical results we consider the following simple biharmonic test problem:

$$\Delta^2 w = f \quad \text{in } \Omega, \qquad w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma$$

on two domains, the square $\Omega = \Omega_S = (-1,1)^2$ and the L-shaped domain $\Omega = \Omega_L$ depicted in figures 6.1 and 6.2, where also the initial mesh (level $\ell = 0$) is shown. The right-hand side $f(x)$ is chosen such that

$$w(x) = \big[1 - \cos(2\pi x_1)\big] \, \big[1 - \cos(4\pi x_2)\big]$$

is the exact solution to the problem. The initial meshes are uniformly refined until the final level $\ell = L$. In all experiments the polynomial degree k as introduced in the beginning of Section 4 is chosen equal to 1, which represents the lowest order HHJ method.

For each of the three multigrid preconditioners we choose one multigrid V-cycle with one forward and one backward Gauss-Seidel sweep for pre- and post-smoothing, respectively. In each of the three steps, a reduction of the Euclidean norm of the initial residual by a factor of 10^{-8} was used as stopping criterion for the PCG methods with initial guess equal to 0.

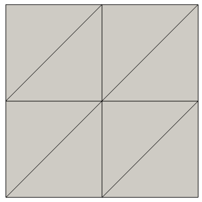

FIG. 6.1. $\Omega = \Omega_S$.

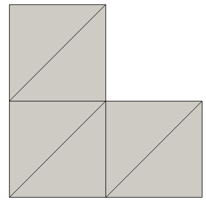

FIG. 6.2. $\Omega = \Omega_L$.

Table 6.1 shows the observed number of iterations for the solution procedure as described above for $\Omega = \Omega_S$. The first column contains the level L of refinement. The next three pairs of columns show the total number N_i of degrees of freedom and the number of iterations iter_i of the PCG method for the linear system in step $i = 1, 2, 3$.

TABLE 6.1
Number of iterations, $\Omega = \Omega_S$ (square).

L	N_1	iter_1	N_2	iter_2	N_3	iter_3
7	64 001	10	132 098	14	64 001	10
8	261 221	10	526 338	15	261 221	10
9	1 046 530	11	2 101 250	15	1 046 530	11
10	4 190 210	11	8 396 802	15	4 190 210	11

Table 6.2 shows the corresponding results for the L-shaped domain $\Omega = \Omega_L$ representing a non-convex case.

TABLE 6.2
Number of iterations, $\Omega = \Omega_L$ (L-shaped domain).

L	N_1	iter_1	N_2	iter_2	N_3	iter_3
7	48 665	11	99 330	16	48 665	11
8	195 585	11	395 266	16	195 585	11
9	784 385	11	1 576 962	16	784 385	11
10	3 141 630	12	6 299 650	17	3 141 630	12

In accordance with well-established convergence results for multigrid methods the number of iterations is bounded uniformly with respect to the mesh size.

Finally, in Table 6.3 the discretization error of the non-conforming method (5.1) and its conforming variant (5.2) are compared. For the non-conforming method, the L^2-error of the original variable w decreases with the order h^2, in accordance with known estimates, see [3, 13]. The conforming variant is more accurate by roughly one digit.

TABLE 6.3
Discretization error $\|w - w_h\|_0$.

L	(5.1)	(5.2)
7	$6.08 * 10^{-4}$	$8.13 * 10^{-5}$
8	$1.52 * 10^{-4}$	$2.03 * 10^{-5}$
9	$3.80 * 10^{-5}$	$5.08 * 10^{-6}$
10	$9.45 * 10^{-6}$	$1.27 * 10^{-7}$

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