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ON THE ROBUSTNESS AND OPTIMALITY OF ALGEBRAIC MULTILEVEL METHODS FOR REACTION-DIFFUSION TYPE PROBLEMS

J. KRAUS AND M. WOLFMAYR

Abstract. This paper is on preconditioners for reaction-diffusion problems that are both, uniform with respect to the reaction- and diffusion coefficients, and optimal in terms of computational complexity. The considered preconditioners belong to the class of so-called algebraic multilevel iteration (AMLI) methods, which are based on a multilevel block factorization and polynomial stabilization. The main focus of this work is on the construction and on the analysis of a hierarchical splitting of the conforming finite element space of piecewise linear functions that allows to meet the optimality conditions for the related AMLI preconditioner in case of second-order elliptic problems with non-vanishing zero-order term. The finite element method (FEM) then leads to a system of linear equations with a system matrix that is a weighted sum of stiffness and mass matrices. Bounds for the constant γ in the strengthened Cauchy-Bunyakowski-Schwarz inequality are computed for both, mass and stiffness matrices in case of a general m-refinement. Moreover, an additive preconditioner is presented for the pivot blocks that arise in the course of the multilevel block factorization. Its optimality is proven for the case $m = 3$. Together with the estimates for γ this shows that the construction of a uniformly convergent AMLI method with optimal complexity is possible (for $m \geq 3$). Finally, we discuss the practical application of this preconditioning technique in the context of time-periodic parabolic optimal control problems.

1. Introduction

This paper is devoted to the construction of efficient and robust preconditioners of optimal complexity for solving reaction-diffusion type problems. Reaction-diffusion problems describe a wide range of physical phenoma. Their finite element (FE) discretization typically leads to a system of linear equations corresponding to a weighted sum of stiffness and mass matrices. Similar problems (on the discrete level) arise from the space-time discretization of parabolic problems, for instance, in modeling unsteady heat conduction. Moreover, these weighted sums of stiffness and mass matrices occur in the context of discretizing time-periodic parabolic optimal control problems by the multiharmonic FEM, see, e.g., [13, 14, 19, 20].

In this paper we study algebraic multilevel iteration (AMLI) preconditioners for systems of linear algebraic equations arising from lowest-order conforming FE approximations of reaction-diffusion type problems. The main goal is to verify their optimality conditions in the classical framework of linear AMLI [5, 6]. The linear AMLI methods have originally been introduced in a multiplicative form, and in the context of hierarchical splittings of conforming FE spaces. Their robustness with respect to general mesh- and/or coefficient anisotropy follows from uniform estimates of the constant γ in the strengthened Cauchy-Bunyakowski-Schwarz inequality associated with a hierarchical splitting of the underlying FE space. Such estimates have first been presented in [23] for linear elements. For quadratic (and higher order) elements the standard hierarchical splittings are in general not robust

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in this respect, cf. [23, 3]. As it was shown in [22] the use of biquadratic (or bicubic) elements in combination with (balanced) semi-coarsening strategies can remedy this problem.

Variable-step AMLI methods that result in nonlinear preconditioners have been introduced in [7] and further analyzed in [15, 24]. In contrast to the linear AMLI the stabilization in nonlinear AMLI is achieved by performing a few inner iterations of a flexible Krylov method on each or on certain levels of the multilevel cycle. More recently the nonlinear AMLI methods have also been combined with additive Schur complement approximations to obtain fully parameter-robust preconditioners for elliptic problems with a highly varying coefficient [16], and problems with a highly anisotropic diffusion tensor [17]. The method proposed in [16] has been applied successfully in [20] to realize a block-diagonal symmetric positive definite preconditioner for the indefinite systems arising in time-periodic parabolic optimal control problems.

Although the nonlinear AMLI methods have considerable advantages from a practical point of view, the focus of the present work is on the construction of optimal *linear* AMLI methods for reaction-diffusion type problems in the (classical) setting of hierarchical bases. We consider a hierarchy of meshes obtained from a recursive regular refinement of an initial coarse triangulation where each triangle is subdivided into m^2 congruent triangles, in the subsequent referred to as m-refinement. The mesh hierarchy provides a natural two-by-two splitting of the system matrix. A quality measure of this splitting is the constant γ in the strengthened Cauchy-Bunyakowski-Schwarz (CBS) inequality. We prove an upper bound for the CBS constant of the mass matrix for a general m-refinement. In [3], the authors have proved an upper bound for the CBS constant of the stiffness matrix for the m-refinement. The combination of these two results yields an estimate for any linear combination of stiffness and mass matrices, which in turn shows that the optimality conditions for the (linear) AMLI preconditioner can be met for $m \geq 3$.

A second important building block for the (classical) AMLI is a uniform preconditioner for the pivot blocks in the recursive two-by-two block factorization of the two-level hierarchical matrices. Here, we generalize the additive preconditioner, which was proposed in [4] for the 2-refinement, for the 3-refinement and prove that its relative condition number is uniformly bounded for arbitrary linear combinations of stiffness and mass matrices. This proof uses the cylindrical algebraic decomposition (CAD), see [10, 11, 12], a standard tool in symbolic computations

The paper is organized as follows: In Section 2, we state the reaction-diffusion model problem, give its variational formulation and its discretization by the finite element method. The construction of the linear AMLI preconditioner for the the system matrix is summarized in Section 3. In Section 4, we present the hierarchical splitting of the system matrices for m-refinements and we prove an upper bound for the related CBS constant of the mass matrix. Section 5 is devoted to the construction of an additive preconditioner of the pivot block in the two-by-two block factorization. A uniform condition number estimate is derived in case of the 3-refinement. In Section 6, we comment on the stabilization polynomials of higher degree. Section 7 presents the application of this AMLI preconditioner to time-periodic parabolic optimal control problems. Finally, some conclusions are drawn in Section 8.

2. A reaction-diffusion type problem

Let $\Omega \subset \mathbb{R}^2$ be a two-dimensional bounded Lipschitz domain with boundary $\Gamma := \partial\Omega$. We consider the following heterogeneous reaction-diffusion problem:

$$\begin{aligned} -\nabla \cdot (\nu(\boldsymbol{x})\nabla u(\boldsymbol{x})) + \mu(\boldsymbol{x})u(\boldsymbol{x}) &= f(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}) &= 0, \qquad \boldsymbol{x} \in \Gamma, \end{aligned} \tag{1}$$

where the coefficients ν and μ are assumed to be measurable, uniformly bounded and to be positive and non-negative, respectively, i.e.,

$$0 < \underline{\nu} \leq \nu(\boldsymbol{x}) \leq \overline{\nu} \quad \text{and} \quad 0 \leq \underline{\mu} \leq \mu(\boldsymbol{x}) \leq \overline{\mu}, \quad \boldsymbol{x} \in \Omega.$$

Usually, these coefficients are piecewise constant, e.g., due to different material parameters in different subdomains.

2.1. The variational problem.

In order to formulate the variational problem corresponding to (1), one multiplies the first equation of (1) by a test function $v \in V$, where V is the Hilbert space

$$V := H_0^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega),\, u = 0 \text{ on } \Gamma\}$$

equipped with the norm $\|\cdot\|_{H^1(\Omega)}$,

$$\|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)},$$

and integrates over Ω. Integration by parts finally yields the following variational problem: Given $f \in L^2(\Omega)$, find $u \in V$ such that

$$\int_\Omega \left[\nu\, \nabla u \cdot \nabla v + \mu\; u\, v\right] d\boldsymbol{x} = \int_\Omega f\, v\, d\boldsymbol{x} \tag{2}$$

for all test functions $v \in V$. Existence and uniqueness of the solution of problem (2) is guaranteed due to the Lax-Milgram lemma.

2.2. The finite element discretization.

In order to solve the reaction-diffusion problem (1), we discretize problem (2) by a conforming FEM. More precisely, we approximate the solution $u \in V$ by a finite element function $u_h \in V_h \subset V$. Let us consider the space V_h to be the final space in a sequence of nested spaces, i.e.,

$$V^{(0)} \subset V^{(1)} \subset ... \subset V^{(k)} \subset ... \subset V^{(K)} = V_h$$

corresponding to a sequence of nested meshes $\mathcal{T}^{(k)}$ for $k = 0, ..., K$ where $\mathcal{T}_h = \mathcal{T}^{(K)}$ is the finest mesh. The spaces

$$V^{(k)} = \operatorname{span}\{\phi_k^{(1)}, ..., \phi_k^{(N^{(k)})}\} \tag{3}$$

are finite element spaces spanned by the standard nodal basis functions $\{\phi_k^{(i)} : i = 1, ..., N^{(k)}\}$. We use continuous, piecewise linear conforming finite elements on triangles on a regular triangulation to construct the finite element spaces and their bases, see [9]. This yields the following linear system arising from the variational formulation (2) on the finest triangulation $\mathcal{T}_h$:

$$\underbrace{(K_{\nu,h} + M_{\mu,h})}_{=:A_h}\, \underline{u}_h = \underline{f}_h, \tag{4}$$

where $K_{\nu,h}$ and $M_{\mu,h}$ correspond to the weighted stiffness and weighted mass matrix, respectively, and $\underline{f}_h$ denotes the load vector. Their entries are computed by the formulas

$$K^{ij}_{\nu,h} = \int_\Omega \nu\, \nabla \phi_K^{(i)} \cdot \nabla \phi_K^{(j)}\, d\boldsymbol{x}, \quad M^{ij}_{\mu,h} = \int_\Omega \mu\, \phi_K^{(i)} \phi_K^{(j)}\, d\boldsymbol{x},$$

$$\underline{f}_h = \left[\int_\Omega f\, \phi_K^{(j)}\, d\boldsymbol{x}\right]_{j=1,...,N_h} \qquad \text{for } i, j = 1, ..., N^{(K)} = N_h.$$

The system (4) has to be solved for the vector $\underline{u}_h = (u_i)_{i=1,\dots,N_h} \in \mathbb{R}^{N_h}$ of nodal unknowns of the finite element approximation

$$u_h(x) = \sum_{i=1}^{N_h} u_i \, \phi_K^{(i)}(\boldsymbol{x}).$$

In order to solve problem (4) efficiently one needs a robust optimal preconditioner. Such a preconditioner can be implemented by various methods such as algebraic multigrid (AMG), domain decomposition (DD) or the AMLI methods. In the following, we will discuss the construction of AMLI preconditioners, which have been introduced in [5] and [6]. The main contribution of this paper is to present a rigorous proof of their robustness and optimal comlexity when used for solving the linear system (4).

3. The AMLI method

In this section, we want to present the main steps in constructing a linear AMLI method for reaction-diffusion type problems. A pseudocode of the linear AMLI algorithm can be found for instance in [6].

3.1. **Two-by-two block factorization.** Let the symmetric and positive definite matrix $A_h = A^{(K)}$ in (4) be obtained in the course of a regular refinement procedure, which defines a sequence of symmetric positive definite matrices starting from a coarsest level system matrix $A^{(0)}$, i.e.,

$$\{A^{(k)}\}, \quad A^{(k)} \in L(\mathbb{R}^{N^{(k)}}, \mathbb{R}^{N^{(k)}}),$$

where $k = 0, ..., K$, and with $N^{(k)} > N^{(k-1)}$, for $k = 1, ..., K$, see [8]. We construct the matrices for the sequence of the nested spaces $V^{(k)}$ corresponding to the nested meshes $\mathcal{T}^{(k)}$. On each level k, we partition the matrix $A^{(k)}$ in a two-by-two block form, i.e.,

$$A^{(k)} = \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{bmatrix} \begin{matrix} \} N^{(k)} - N^{(k-1)} \\ \} N^{(k-1)} \end{matrix} . \tag{5}$$

The Schur complements $S^{(k)} = A_{22}^{(k)} - A_{21}^{(k)} {A_{11}^{(k)}}^{-1} A_{12}^{(k)}$ are dense symmetric positive definite (SPD) matrices. In the course of designing optimal multilevel methods it is important to construct a sparse approximation of $S^{(k)}$, see [18]. More precisely, we want $S^{(k)}$ to be spectrally equivalent to $A^{(k-1)}$ on all levels with spectral equivalence bounds that neither depend on the level index $k-1$ nor on any problem parameters, see [8].

3.2. **Estimates of the CBS constant.** The efficiency of preconditioners based on two-by-two block factorization strongly depends on the coupling of the diagonal blocks of the two-level matrix via its off-diagonal blocks. A measure for the strength of this coupling is the constant γ in the strengthened Cauchy-Bunyakowski-Schwarz (CBS) inequality

$$\left| \mathbf{v}_1^T A_{12}^{(k)} \mathbf{v}_2 \right| \leq \gamma \left(\mathbf{v}_1^T A_{11}^{(k)} \mathbf{v}_1 \, \mathbf{v}_2^T A_{22}^{(k)} \mathbf{v}_2 \right)^{1/2}, \tag{6}$$

where $\mathbf{v}_1 \in V_1^{(k)}$, $\mathbf{v}_2 \in V_2^{(k)}$ and $V_1^{(k)} \times V_2^{(k)}$ is a splitting of the vector space $V^{(k)}$, which is consistent with the partioning (5). The CBS constant, i.e., the smallest constant for which (6) holds for all $\mathbf{v} \in V^{(k)}$, can be estimated locally, see, e.g., [21] and the references therein. The global CBS constant then can be estimated by the maximum of the local CBS constants on the macroelements $E^{(k)} \subset \mathcal{T}^{(k)}$, i.e.,

$$\gamma \leq \max_{E^{(k)} \subset \mathcal{T}^{(k)}} \gamma_{E^{(k)}} \leq 1. \tag{7}$$

The macroelements $E^{(k)} \subset \mathcal{T}^{(k)}$ are uniform refinements of the coarse-grid elements $e^{(k-1)} \in \mathcal{T}^{(k-1)}$. The local CBS constant $\gamma_{E^{(k)}}$ can be computed via a simple rule, i.e.,

$$\gamma^2_{E^{(k)}} = 1 - \lambda^{\min}_E, \tag{8}$$

where $\lambda^{\min}_E$ is the minimal eigenvalue of the generalized eigenvalue problem

$$S_{E^{(k)}} \mathbf{v}_{E^{(k)}:2} = \lambda\, A_{e^{(k-1)}} \mathbf{v}_{E^{(k)}:2}, \tag{9}$$

where $\mathbf{v}_{E^{(k)}:2} \neq (c, c, ..., c)^T$, c is a real constant, see, e.g., [18]. The global matrices $A^{(k)}$ and $A^{(k-1)}$ can be computed via the local matrices $A_{E^{(k)}}$ and $A_{e^{(k-1)}}$, respectively. The standard FEM assembling can be written in the form

$$A^{(k)} = \sum_{E^{(k)} \subset \mathcal{T}^{(k)}} R^T_{E^{(k)}} A_{E^{(k)}} R_{E^{(k)}}, \tag{10}$$

$$A^{(k-1)} = \sum_{e^{(k-1)} \in \mathcal{T}^{(k-1)}} R^T_{e^{(k-1)}} A_{e^{(k-1)}} R_{e^{(k-1)}}, \tag{11}$$

where $R_{E^{(k)}}$ and $R_{e^{(k-1)}}$ are is the restriction mappings of a global vector of unknowns at levels k and $k-1$ to the local vectors corresponding to the (macro)element $E^{(k)} \subset \mathcal{T}^{(k)}$, and $e^{(k-1)} \in \mathcal{T}^{(k-1)}$, respectively, cf. [18]. Hence, it suffices to consider the local matrices $A_{E^{(k)}}$ and $A_{e^{(k-1)}}$ for analyzing the robustness and optimal comlexity of the linear AMLI method for solving problem (4).

The parameters ν and μ of problem (4) are assumed to be constant on the coarsest mesh partitioning $\mathcal{T}^{(0)}$. Then the system matrix corresponding to the coarsest mesh can be written as

$$\begin{aligned} A^{(0)} &= \sum_{e^{(0)} \in \mathcal{T}^{(0)}} R^T_{e^{(0)}} A_{e^{(0)}} R_{e^{(0)}} \\ &= \sum_{e^{(0)} \in \mathcal{T}^{(0)}} R^T_{e^{(0)}} \left(\nu_{e^{(0)}} K_{e^{(0)}} + \mu_{e^{(0)}} M_{e^{(0)}}\right) R_{e^{(0)}} \\ &= \sum_{e^{(0)} \in \mathcal{T}^{(0)}} \nu_{e^{(0)}} R^T_{e^{(0)}} \left(K_{e^{(0)}} + \tilde{\mu}_{e^{(0)}} M_{e^{(0)}}\right) R_{e^{(0)}}, \end{aligned} \tag{12}$$

where $\tilde{\mu}_{e^{(0)}} = \mu_{e^{(0)}}/\nu_{e^{(0)}} \geq 0$ and $\nu_{e^{(0)}} > 0$.

3.3. **Hierarchical splitting and stabilization techniques.** Let us consider the two nested finite element spaces $V^{(k-1)} \subset V^{(k)}$ which correspond to the two consecutive regular mesh refinements $\mathcal{T}^{(k-1)}$ and $\mathcal{T}^{(k)}$, respectively. Their standard FE nodal basis functions are given by $\{\phi^{(i)}_{k-1}, i = 1, ..., N^{(k-1)}\}$ and $\{\phi^{(i)}_k, i = 1, ..., N^{(k)}\}$. We split the $N^{(k)}$ meshpoints into the group containing the $N^{(k-1)}$ nodes of the coarse mesh $\mathcal{T}^{(k-1)}$ and the rest. Then by defining the hierarchical basis functions $\{\tilde{\phi}^{(i)}_k, i = 1, ..., N^{(k)}\}$, the hierarchical matrix $\tilde{A}^{(k)}$ as well as $A^{(k)}$ (for the latter see (5)) are naturally partitioned in a two-by-two block form, i.e.,

$$\tilde{A}^{(k)} = \begin{bmatrix} \tilde{A}^{(k)}_{11} & \tilde{A}^{(k)}_{12} \\ \tilde{A}^{(k)}_{21} & \tilde{A}^{(k)}_{22} \end{bmatrix} \begin{matrix} \}N^{(k)} - N^{(k-1)} \\ \}N^{(k-1)} \end{matrix}, \tag{13}$$

see [18]. The hierarchical matrix $\tilde{A}^{(k)}$ is more dense than $A^{(k)}$. However, the nodal unknown vectors for the standard and for the hierarchical basis functions are related by a transformation matrix of the form

$$J^{(k)} = \begin{bmatrix} I & J^{(k)}_{12} \\ 0 & I \end{bmatrix}, \tag{14}$$

where I is the identity matrix and 0 the zero matrix. In practical applications, we can work with $A^{(k)}$ instead of $\tilde{A}^{(k)}$, since

$$\tilde{A}^{(k)} = \left(J^{(k)}\right)^T A^{(k)} J^{(k)}, \tag{15}$$

and the basis transformation does not change the Schur complement, i.e., $S_{E^{(k)}} = \tilde{S}_{E^{(k)}}$. So we can compute the minimal eigenvalue of the generalized eigenvalue problem (9) using the Schur complement $S_{E^{(k)}}$, from which we obtain the local CBS constant $\gamma_{E^{(k)}}$ via the rule (8).

In order to construct uniform AMLI preconditioners whose application to a vector has optimal computational complexity, we combine hierarchical basis preconditioners with polynomial stabilization techniques, see [5, 6]. The stabilization polynomials $P^{(k)}(t) = P_{\upsilon_k}(t)$ of degree υ_k have to satisfy the condition $P^{(k)}(0) = 1$ for all $k = 1, \ldots, K$. They are used for computing polynomial approximations of the inverse of the Schur complement $S^{(k)}$. Some details regarding a proper choice of stabilization polynomials are provided in Section 6.

The optimality conditions for the multiplicative variant of the AMLI method are given by

$$\frac{1}{\sqrt{1-\gamma^2}} < \upsilon < \varrho. \tag{16}$$

For additive AMLI preconditioner the optimality conditions read

$$\sqrt{\frac{1+\gamma}{1-\gamma}} < \upsilon < \varrho. \tag{17}$$

Here υ is the degree of the stabilization polynomial and the parameter ϱ stands for the refinement factor, see [18] and the references therein. More precisely, in case of an m-refinement we have $\varrho = m^2$, which means that we subdivide one element into m congruent elements in one refinement step.

4. Robust AMLI algorithms for conforming linear finite elements

Since the system matrices on the coarsest level satisfy the assembling property (12), we consider the element system matrix $A_{e^{(0)}} = K_{e^{(0)}} + \tilde{\mu}_{e^{(0)}} M_{e^{(0)}}$ for an arbitrary element $e^{(0)} \in \mathcal{T}^{(0)}$, which is a weighted sum of stiffness and mass matrices. Due to the standard FEM assembling (10) the analysis of uniform local bounds has to be carried out for a (macro) element matrix corresponding to an arbitrary triangle.

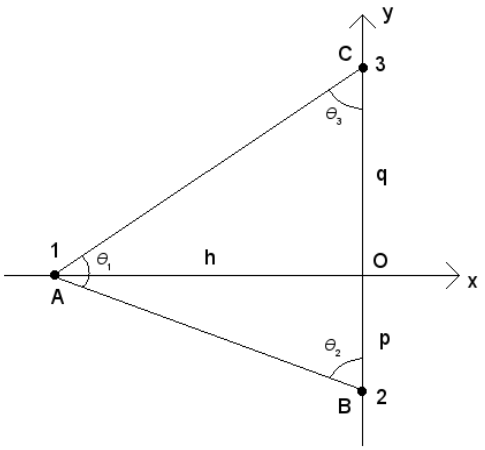

Figure 1. An arbitrary non-degenerate triangle $e \in \mathcal{T}^{(k)}$.

The element stiffness matrix K_e for the Laplace operator is given by

$$K_e = \frac{1}{2}\begin{pmatrix} b+c & -c & -b \\ -c & a+c & -a \\ -b & -a & a+b \end{pmatrix}, \tag{18}$$

where a, b and c are equal to the cotangens of the angles in the triangle e. The proof of (18) can be found in [2, 23], see also [18].

Lemma 1. *The element mass matrix M_e can be written in the general form*

$$M_e = \frac{h^2(b+c)}{24}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \tag{19}$$

where a, b and c are equal to the cotangens of the angles in the triangle e, i.e., $a = \cot\theta_1$, $b = \cot\theta_2$, $c = \cot\theta_3$, and h is the triangle height measured perpendicular to the side BC, where B and C are the vertices with the angles θ_2 and θ_3, respectively.

Proof. The element mass matrix for a given arbitrary non-degenerate triangle e is given by

$$M_e(u,v) = \int_e uv\, de.$$

We introduce the notations $h = |OA|$, $p = |OB|$ and $q = |OC|$, where O is the origin, see Fig. 1. Then we have the following relations given:

$$b = \frac{p}{h}, \quad c = \frac{q}{h}, \quad a = \cot(\pi - (\theta_2 + \theta_3)) = \frac{h^2 - pq}{h(p+q)}.$$

The element basis functions are given by

$$\phi_1 = -\frac{x}{h}, \;\; \phi_2 = \frac{qx + h(q-y)}{h(p+q)}, \;\; \phi_3 = \frac{px + h(p+y)}{h(p+q)}.$$

Moreover,

$$|e| = \int de = \frac{h(p+q)}{2} = J_e \cdot \frac{1}{2},$$

where $J_e = h^2(b+c)$ is the Jacobi determinant. We obtain the following first two entries of the element mass matrix:

$$M_{e11} = \int_{-h}^{0}\int_{-p/hx-p}^{q/hx+q} (\phi_1)^2\, dy\, dx = \frac{h^2(b+c)}{12}$$

and

$$M_{e12} = \int_{-h}^{0}\int_{-p/hx-p}^{q/hx+q} \phi_1\phi_2\, dy\, dx = \frac{h^2(b+c)}{24}.$$

Analogously, we obtain all other entries and we finally derive the element mass matrix (19). □

We assume without loss of generality that $|a| \le b \le c$. Moreover, we define $\alpha = a/c$ and $\beta = b/c$ and obtain the following representations for the element stiffness and mass matrices:

$$K_e = \frac{c}{2}\begin{pmatrix} \beta+1 & -1 & -\beta \\ -1 & \alpha+1 & -\alpha \\ -\beta & -\alpha & \alpha+\beta \end{pmatrix} \quad \text{and} \quad M_e = \frac{h^2 c\,(\beta+1)}{24}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

where $(\alpha, \beta) \in D$ with

$$D = \{(\alpha,\beta) \in \mathbb{R}^2 : -\frac{1}{2} < \alpha \le 1, \max\{-\frac{\alpha}{\alpha+1}, |\alpha|\} \le \beta \le 1\}, \tag{20}$$

see [18] and the references therein.

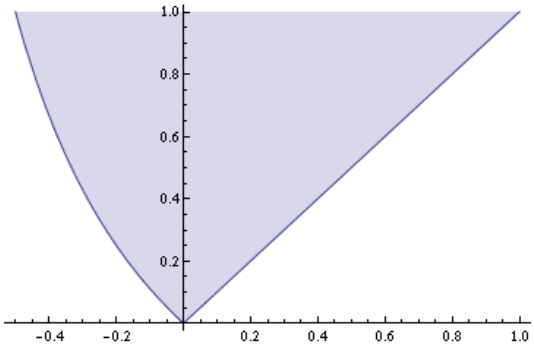

FIGURE 2. Domain D of the parameters α and β.

In case of discretizing diffusion problems by conforming linear finite elements, the standard choice is a (uniform) 2-refinement, which means that each coarse element is subdivided into four congruent elements in every refinement step. In this paper, we consider the general case of an m-refinement, where each element is subdivided into m^2 elements in every refinement step. In the next subsection, we will give the reason why a 2-refinement in general is not sufficient for problems of the form (4).

4.1. **The 2-refinement.** We consider the element stiffness matrix (18) for an arbitrary element $e \in \mathcal{T}^{(k-1)}$. On the related macro element $E \subset \mathcal{T}^{(k)}$, which is

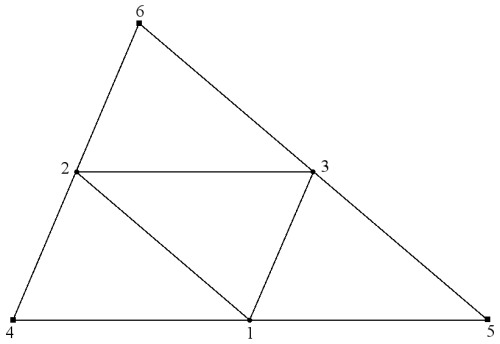

FIGURE 3. Subdivision of one triangle into four congruent ones.

obtained by subdivision of the coarse triangle into four congruent triangles, see Fig. 3, we obtain the following stiffness matrix:

$$K_E = \begin{pmatrix} a+b+c & -a & -b & -\frac{c}{2} & -\frac{c}{2} & 0 \\ -a & a+b+c & -c & -\frac{b}{2} & 0 & -\frac{b}{2} \\ -b & -c & a+b+c & 0 & -\frac{a}{2} & -\frac{a}{2} \\ -\frac{c}{2} & -\frac{b}{2} & 0 & \frac{b+c}{2} & 0 & 0 \\ -\frac{c}{2} & 0 & -\frac{a}{2} & 0 & \frac{a+c}{2} & 0 \\ 0 & -\frac{b}{2} & -\frac{a}{2} & 0 & 0 & \frac{a+b}{2} \end{pmatrix}.$$

The hierarchical stiffness matrix is given by

$$\tilde{K}_E = J^T K_E J = \begin{pmatrix} \tilde{K}_{E:11} & \tilde{K}_{E:12} \\ \tilde{K}_{E:21} & \tilde{K}_{E:22} \end{pmatrix}$$

with

$$J = \begin{pmatrix} I & J_{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{K}_{E:11} = \begin{pmatrix} a+b+c & -a & -b \\ -a & a+b+c & -c \\ -b & -c & a+b+c \end{pmatrix},$$

$$\tilde{K}_{E:12} = -\frac{1}{2} \begin{pmatrix} -b & -a & a+b \\ -c & a+c & -a \\ b+c & -c & b \end{pmatrix},$$

$$\tilde{K}_{E:21} = (\tilde{K}_{E:12})^T, \quad \text{and} \quad \tilde{K}_{E:22} = K_e.$$

In [23], the authors have proved the following explicit formula for the local CBS constant $\gamma_{K,E}$ in case of a 2-refinement:

$$\gamma_{K,E}^2 = \frac{3}{8} + \frac{1}{4}\sqrt{\sum_{i=1}^{3} \cos^2\theta_i - \frac{3}{4}}. \tag{21}$$

One way to derive this result is to solve the generalized eigenvalue problem (9) and then to compute $\gamma_{K,E}$ via the rule (8). From (21) follows that $\gamma_{K,E}^2 = 3/4$ in the worst case.

Analogously, we can compute the CBS constant for the mass matrix. The mass matrix on a macro element $E \subset \mathcal{T}^{(k)}$ is given by

$$M_E = \frac{h^2(b+c)}{24} \begin{pmatrix} 6 & 2 & 2 & 1 & 1 & 0 \\ 2 & 6 & 2 & 1 & 0 & 1 \\ 2 & 2 & 6 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 2 \end{pmatrix}$$

and the hierarchical mass matrix by

$$\tilde{M}_E = J^T M_E J = \begin{pmatrix} \tilde{M}_{E:11} & \tilde{M}_{E:12} \\ \tilde{M}_{E:21} & \tilde{M}_{E:22} \end{pmatrix},$$

where

$$\tilde{M}_{E:11} = \frac{h^2(b+c)}{24} \begin{pmatrix} 6 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{pmatrix}, \quad \tilde{M}_{E:12} = (\tilde{M}_{E:21})^T = \frac{h^2(b+c)}{24} \begin{pmatrix} 5 & 5 & 2 \\ 5 & 2 & 5 \\ 2 & 5 & 5 \end{pmatrix}$$

and $\tilde{M}_{E:22} = 4\, M_e$. We solve the generalized eigenvalue problem

$$S_M\, \mathbf{v}_{E:2} = \lambda\, \tilde{M}_{E:22}\, \mathbf{v}_{E:2}, \tag{22}$$

where $S_M = M_{E:22} - M_{E:21} M_{E:11}^{-1} M_{E:12} = \tilde{S}_M$ is the Schur complement for the mass matrix, and obtain the eigenvalue $7/16$ twice and the eigenvalue $1/10$ once (independent of all parameters a, b, c!). The local CBS constant for the mass matrix therefore is given by

$$\gamma_{M,E} = \sqrt{1 - \min\left\{\frac{7}{16}, \frac{1}{10}\right\}} = \sqrt{1 - \frac{1}{10}} = \sqrt{\frac{9}{10}}.$$

Hence, we obtain the following estimate for the CBS constant of the weighted sum $A_E = K_E + \tilde{\mu}_E M_E$:

$$\gamma_{A,E}^2 \leq \max\left\{\gamma_{K,E}^2, \gamma_{M,E}^2\right\} = \max\left\{\frac{3}{4}, \frac{9}{10}\right\} = \frac{9}{10}. \tag{23}$$

We observe that the estimate (23) does not imply the optimality condition (16) because there is no integer degree υ of the stabilization polynomial satisfying

$$\frac{1}{\sqrt{1-9/10}} = \sqrt{10} \approx 3.16228 < \upsilon < \varrho = m^2 = 4. \tag{24}$$

This is the reason to consider m-refinements for $m > 2$. In the next subsection, we analyze the case $m = 3$.

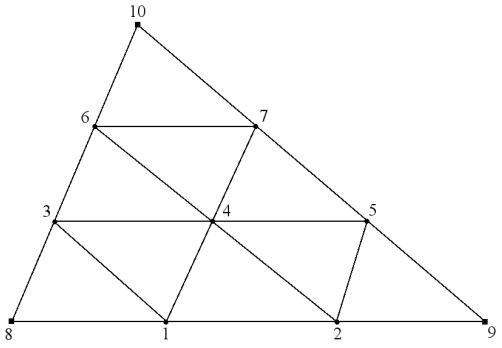

FIGURE 4. Subdivision of one triangle into nine congruent ones.

4.2. **The 3-refinement.** In case of a 3-refinement, see Fig. 4, one macro element $E \subset \mathcal{T}^{(k)}$ is subdived into nine congruent triangles. The corresponding stiffness matrix K_E and its hierarchical stiffness matrix $\tilde{K}_E$ on the macro element are then given by

$$K_E = \begin{pmatrix} K_{E:11} & K_{E:12} \\ K_{E:21} & K_{E:22} \end{pmatrix} \text{ and } \tilde{K}_E = J^T K_E J,$$

where

$$K_{E:11} = \begin{pmatrix} s & -\frac{c}{2} & -a & -b & 0 & 0 & 0 \\ -\frac{c}{2} & s & 0 & -a & -b & 0 & 0 \\ -a & 0 & s & -c & 0 & -\frac{b}{2} & 0 \\ -b & -a & -c & 2s & -c & -a & -b \\ 0 & -b & 0 & -c & s & 0 & -\frac{a}{2} \\ 0 & 0 & -\frac{b}{2} & -a & 0 & s & -c \\ 0 & 0 & 0 & -b & -\frac{a}{2} & -c & s \end{pmatrix}, s = a+b+c,$$

$$K_{E:12} = (K_{E:21})^T = \begin{pmatrix} -\frac{c}{2} & 0 & 0 \\ 0 & -\frac{c}{2} & 0 \\ -\frac{b}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{a}{2} & 0 \\ 0 & 0 & -\frac{b}{2} \\ 0 & 0 & -\frac{a}{2} \end{pmatrix}, \quad K_{E:22} = \begin{pmatrix} \frac{b+c}{2} & 0 & 0 \\ 0 & \frac{a+c}{2} & 0 \\ 0 & 0 & \frac{a+b}{2} \end{pmatrix},$$

with the transformation matrix

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \tilde{K}_{E:11} = K_{E:11},$$

$$\tilde{K}_{E:12} = (\tilde{K}_{E:21})^T = -\frac{1}{3} \begin{pmatrix} -b & -a & a+b \\ -b & -a & a+b \\ -c & a+c & -a \\ 0 & 0 & 0 \\ b+c & -c & -b \\ -c & a+c & -a \\ b+c & -c & -b \end{pmatrix}$$

and $\tilde{K}_{E:22} = K_e$. In [3], the authors proved the following estimate of the CBS constant of a macro element stiffness matrix arising from a uniform m-refinement:

$$\gamma_{K,E}^2 \leq \frac{m^2-1}{m^2}. \tag{25}$$

Hence, for $m = 3$, we have the following estimate: $\gamma_E^2 \leq 8/9$.

Now, we estimate the CBS constant for the corresponding macroelement mass matrix, where we use the node numbering shown in Fig. 4. First we find

$$M_E = \frac{h^2(b+c)}{24} \begin{pmatrix} 6 & 1 & 2 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 6 & 0 & 2 & 2 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 6 & 2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 2 & 2 & 12 & 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 6 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 6 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 2 & 6 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 \end{pmatrix}$$

and the hierarchical macroelement mass matrix

$$\tilde{M}_E = J^T M_E J = \begin{pmatrix} \tilde{M}_{E:11} & \tilde{M}_{E:12} \\ \tilde{M}_{E:21} & \tilde{M}_{E:22} \end{pmatrix},$$

where $\tilde{M}_{E:11} = M_{E:11}$, $\tilde{M}_{E:22} = 9\, M_e$ and

$$\tilde{M}_{E:12} = (\tilde{M}_{E:21})^T = \frac{h^2(b+c)}{24} \begin{pmatrix} \frac{22}{3} & \frac{10}{3} & \frac{4}{3} \\ \frac{10}{3} & \frac{22}{3} & \frac{4}{3} \\ \frac{22}{3} & \frac{4}{3} & \frac{10}{3} \\ 8 & 8 & 8 \\ \frac{4}{3} & \frac{22}{3} & \frac{10}{3} \\ \frac{10}{3} & \frac{4}{3} & \frac{22}{3} \\ \frac{4}{3} & \frac{10}{3} & \frac{22}{3} \end{pmatrix}.$$

We solve the eigenvalue problem (22) corresponding to the 3-refinement and obtain the following eigenvalues which are again independent of all parameters a, b, c: the

eigenvalue 19/99 twice and the eigenvalue 1/21 once. Hence, the local CBS constant for the mass matrix is given by

$$\gamma_{M,E} = \sqrt{1 - \min\left\{\frac{19}{99}, \frac{1}{21}\right\}} = \sqrt{\frac{20}{21}},$$

and, finally, in view of (25), the CBS constant of the weighted sum $A_E = K_E + \tilde{\mu}_E\, M_E$ can be estimated by

$$\gamma_{A,E}^2 \leq \max\left\{\gamma_{K,E}^2, \gamma_{M,E}^2\right\} = \max\left\{\frac{8}{9}, \frac{20}{21}\right\} = \frac{20}{21}. \tag{26}$$

From (26) we conclude that the optimality condition (16) is fulfilled since

$$\frac{1}{\sqrt{1 - 20/21}} = \sqrt{21} \approx 4.58258 < \upsilon < \varrho = m^2 = 9 \tag{27}$$

holds for polynomial degree $\upsilon \in \{5, 6, 7, 8\}$.

In the following subsection, we discuss the estimation of the CBS constant for uniform m-refinements for all $m > 2$. In particular, we present a uniform estimate of the CBS constant for $A_E = K_E + \tilde{\mu}_E\, M_E$ on a macro element $E \subset \mathcal{T}^{(k)}$.

4.3. **The uniform m-refinement.** The CBS constant for the mass matrix for a refinement factor of $\varrho = m^2$ can be computed in the same way as it is presented in Subsection 4.1 and Subsection 4.2. In Theorem 1, we present a general result for the estimation of the CBS constant of the mass matrix.

Theorem 1. *Consider a uniform m-refinement for conforming linear finite elements where $m > 2$. The CBS constant of the mass matrix can be estimated as follows:*

$$\gamma_M^2 \leq \frac{12\, m^2 - 5}{12\, m^2}. \tag{28}$$

Proof. Let $m_c = h^2(b + c)/24$. The global CBS constant can be estimated by the the maximum of the local CBS constants on the macroelements (7), which can be again computed via the rule (8), i.e., $\gamma_{M,E}^2 = 1 - \lambda_E^{\min}$, where $\lambda_E^{\min}$ is the minimal eigenvalue of the eigenvalue problem (22), which we write in the form

$$\mathbf{v}_{E:2}^T \left(S_M - \lambda\, \tilde{M}_{E:22}\right) \mathbf{v}_{E:2} = 0,$$

We try to find a lower bound $\underline{\lambda_E^{\min}}$ for the minimal eigenvalue $\lambda_E^{\min}$ such that

$$\mathbf{v}_{E:2}^T \left(S_M - \underline{\lambda_E^{\min}}\, \tilde{M}_{E:22}\right) \mathbf{v}_{E:2} \geq 0.$$

For that reason, we estimate the Schur complement S_M from below by $\underline{S_M}$, and, after that we solve the problem $\mathbf{v}_{E:2}^T(\underline{S_M} - \lambda\, \tilde{M}_{E:22})\mathbf{v}_{E:2} = 0$. For every m-refinement, we systematically use a bottom-up lexicographical ordering for the fine nodes and number the three coarse nodes last. Hence, the lower right block $M_{E:22}$ of the macroelement mass matrix is always $M_{E:22} = 2\, m_c\, I$, where I is the identity matrix. Let N_E denote the number of nodes on a macro element subdived into m^2 elements. The Schur complement S_M can be estimated from below in the following way:

$$\begin{aligned} S_M &= M_{E:22} - M_{E:21}\, M_{E:11}^{-1}\, M_{E:12} = 2\, m_c\, I - M_{E:21}\, M_{E:11}^{-1}\, M_{E:12} \\ &\geq 2\, m_c\, I - \frac{1}{6}\, m_c^{-1}\, M_{E:21}\, I\, M_{E:12}, \end{aligned}$$

where we use the estimate

$$M_{E:11} \geq \min_{i \in \{1, \ldots, N_E\}} (M_{E:11})_{ii}\, I = 6\, m_c\, I$$

because the weakly diagonally dominant matrix $M_{E:11}$ has only the diagonal entries $12\,m_c$ and $6\,m_c$. Moreover, since the matrices $M_{E:12}$ and $M_{E:21} = (M_{E:12})^T$ have exactly two entries equal to one in each of the three columns and rows, respecively, we obtain

$$M_{E:21}\,M_{E:12} = (M_{E:12})^T\,M_{E:12} = 2\,m_c^2\,I$$

and finally the Schur complement S_M can be estimated from below by

$$S_M \geq 2\,m_c\,I - \frac{1}{6}\,m_c^{-1}\,2\,m_c^2\,I = 2\,m_c\,I - \frac{1}{3}\,m_c\,I = \frac{5}{3}\,m_c\,I.$$

Next we have that $\tilde{M}_{E:22} = M_{E:22} + J_{E:12}^T\,M_{E:12} + M_{E:21}\,J_{E:12} + J_{E:12}^T\,M_{E:12}\,J_{E:12} = m^2\,M_e$ with a transformation matrix $J_{E:12}$ of the from (14). Then

$$J_{E:12}^T\,M_{E:12} = m_c \begin{pmatrix} \frac{2\,m-2}{m} & \frac{1}{m} & \frac{1}{m} \\ \frac{1}{m} & \frac{2\,m-2}{m} & \frac{1}{m} \\ \frac{1}{m} & \frac{1}{m} & \frac{2\,m-2}{m} \end{pmatrix}$$

because $M_{E:12}$ has exactly two entries with value one in each row and $J_{E:12}$ has the value $(m-1)/m$ in exactly the same positions. Hence, we solve the problem

$$\mathbf{v}_{E:2}^T \left(\frac{5}{3}\,m_c\,I - \lambda\,m^2\,M_e \right) \mathbf{v}_{E:2} = 0$$

and obtain the eigenvalues $5/(3m^2)$ twice and $5/(12m^2)$ once. The minimal eigenvalue is $5/(12m^2)$ which yields a lower bound for $\lambda_E^{\min}$. According to (8) we obtain the bound

$$\gamma_{M,E}^2 = 1 - \lambda_E^{\min} \leq 1 - \frac{5}{12\,m^2} = \frac{12\,m^2 - 5}{12\,m^2}.$$

which yields together with (7) the upper bound (28). □

Remark 1. *If the parameter $\tilde{\mu}_E = 0$, i.e., $A_E = K_E$, then the CBS constant can be estimated by the formula (25), see* [3]. *Estimate (28) provides at the same time also a general estimate for the local CBS constant $\gamma_{A,E}$ corresponding to the weighted sum of mass and stiffness matrix, since together with (25) we obtain*

$$\begin{aligned} \gamma_{A,E} &\leq \max\left\{\gamma_{K,E}, \gamma_{M,E}\right\} \\ &\leq \max\left\{ \sqrt{\frac{m^2-1}{m^2}}, \sqrt{\frac{12\,m^2-5}{12\,m^2}} \right\} = \sqrt{\frac{12\,m^2-5}{12\,m^2}}. \end{aligned} \tag{29}$$

Remark 2. *By applying estimate (28) of Theorem 1 and using (25), we obtain the following estimates for the CBS constants corresponding to m-refinements for $m = 3, 4, 5$:*

$$\gamma_{A,E}^2 \leq \max\left\{\gamma_{K,E}^2, \gamma_{M,E}^2\right\} \leq \begin{cases} \max\left\{\frac{8}{9}, \frac{103}{108}\right\} \approx 0.953704 & \text{for } m = 3, \\ \max\left\{\frac{15}{16}, \frac{187}{192}\right\} \approx 0.973958 & \text{for } m = 4, \\ \max\left\{\frac{24}{25}, \frac{59}{60}\right\} \approx 0.983333 & \text{for } m = 5, \end{cases} \tag{30}$$

In comparison to that, we want to compute the sharp bounds for the CBS constants up to a 5-refinement. The cases $m = 2$ and $m = 3$ have already been worked out in Subsections 4.1 and 4.2. In an analogous manner, one computes the estimates for the cases $m = 4$ and $m = 5$ using again a bottom-up lexicographical ordering for the fine nodes and numbering the three coarse nodes last.

For $m = 4$, we obtain the following block J_{12} of the transformation matrix J:

$$J_{12} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 0 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} \end{pmatrix}^T.$$

The resulting hierarchical macroelement mass matrix is given by

$$\tilde{M}_E = J^T M_E J = \begin{pmatrix} \tilde{M}_{E:11} & \tilde{M}_{E:12} \\ \tilde{M}_{E:21} & \tilde{M}_{E:22} \end{pmatrix}$$

where

$$\tilde{M}_{E:11} = \frac{h^2(b+c)}{24} \begin{pmatrix} 6 & 1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 2 & 12 & 2 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 & 12 & 2 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 & 6 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 6 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & 12 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 2 & 6 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 6 \end{pmatrix},$$

$\tilde{M}_{E:22} = 16\, M_e$, $\tilde{M}_{E:12} = (\tilde{M}_{E:21})^T = \frac{h^2(b+c)}{24}\tilde{N}$, *and*

$$\tilde{N} = \begin{pmatrix} \frac{17}{2} & \frac{11}{2} & \frac{5}{2} & \frac{17}{2} & 12 & 6 & 1 & \frac{11}{2} & 6 & 1 & \frac{5}{2} & 1 \\ \frac{5}{2} & \frac{11}{2} & \frac{17}{2} & 1 & 6 & 12 & \frac{17}{2} & 1 & 6 & \frac{11}{2} & 1 & \frac{5}{2} \\ 1 & 1 & 1 & \frac{5}{2} & 6 & 6 & \frac{5}{2} & \frac{11}{2} & 12 & \frac{11}{2} & \frac{17}{2} & \frac{17}{2} \end{pmatrix}^T.$$

Analogously, one determines the transformation matrix J, the macroelement mass matrix M_E and the hierarchical mass matrix $\tilde{M}_E$ for the 5-refinement. Finally, one solves the corresponding eigenvalue problem (22) for $m = 4, 5$, and uses the minimal eigenvalues to compute the CBS constants via the rule (8). The resulting sharp estimates of the CBS constants for problems of the form (4) corresponding to m-refinements for $m \in \{2, 3, 4, 5\}$ (under the assumption (12)) are as follows:

$$\gamma^2_{A,E} \le \max\left\{\gamma^2_{K,E}, \gamma^2_{M,E}\right\} = \begin{cases} \max\left\{\frac{3}{4}, \frac{9}{10}\right\} = 0.9 & \text{for } m = 2, \\ \max\left\{\frac{8}{9}, \frac{20}{21}\right\} \approx 0.952381 & \text{for } m = 3, \\ \max\left\{\frac{15}{16}, \frac{36}{37}\right\} \approx 0.972973 & \text{for } m = 4, \\ \max\left\{\frac{24}{25}, \frac{11916}{12125}\right\} \approx 0.982763 & \text{for } m = 5. \end{cases} \tag{31}$$

Comparing now (30) and (31) shows that formula (28) provides a very good estimate for the CBS constant of the macro element mass matrix.

Summarizing our findings, the smallest value for m that guarantees that the optimality conditions can be satisfied with an m-refinement is $m = 3$. Moreover, since

$$\frac{1}{\sqrt{1 - \frac{12m^2-5}{12m^2}}} = 2m\sqrt{\frac{3}{5}} < 2m < m^2 \quad \text{for all } m > 2,$$

any m-refinement for $m > 2$ allows to meet the optimality conditions (16).

In the next section, we will present the construction and the analysis of an additive preconditioner for the pivot block A_{11} coming from a 3-refinement.

5. Additive preconditioning of the pivot block

Applying the AMLI method requires the action of (an approximation of) the inverse of the pivot blocks $A_{11}^{(k)}$ on a vector. It is well known (see, e.g., [6]) that the (linear) AMLI preconditioner with approximate pivot block $C_{11}^{(k)}$ is optimal if

apart from the optimality conditions (16) the preconditioners $C_{11}^{(k)}$ are spectrally equivalent to $A_{11}^{(k)}$ on all levels k, i.e., $C_{11}^{(k)} \approx A_{11}^{(k)}$, and their action on a vector has linear complexity, i.e., requires $\mathcal{O}(N^{(k)})$ arithmetic operations.

Here we generalize the additive preconditioner C_{11}, which was proposed in [4] for the 2-refinement, for the 3-refinement and derive the corresponding condition number bounds. The construction as well as the analysis of C_{11} relies on a macroelement-by-macroelement assembling procedure again, i.e.,

$$A_{11} = \sum_{E \in \mathcal{T}^{(k-1)}} R_E^T \, A_{E:11} \, R_E \tag{32}$$

and

$$C_{11} = \sum_{E \in \mathcal{T}^{(k-1)}} R_E^T \, C_{E:11} \, R_E. \tag{33}$$

The pivot block of the macro element matrices is given by $A_{E:11} = K_{E:11} + \tilde{\mu}_E \, M_{E:11}$. The idea is to construct an additive preconditioner $C_{E:11}$ having the form $C_{E:11} = C_{E:11}^K + \tilde{\mu}_E \, C_{E:11}^M$ with the same weighting as the pivot block and where the matrices $C_{E:11}^K$ and $C_{E:11}^M$ have the same structure, i.e., the same non-zero pattern, in order to implement the preconditioner $C_{E:11}$. We obtain the preconditioners $C_{E:11}^K$ and $C_{E:11}^M$ by preserving the largest (in magnitude) off-diagonal entries of $K_{E:11}$ and $M_{E:11}$, respectively. Note that the same nonzero pattern is chosen for the preconditioner of the stiffness matrix pivot block and for the one of the mass matrix pivot block! The couplings corresponding to the largest (in magnitude) off-diagonal entries are shown in Fig. 5.

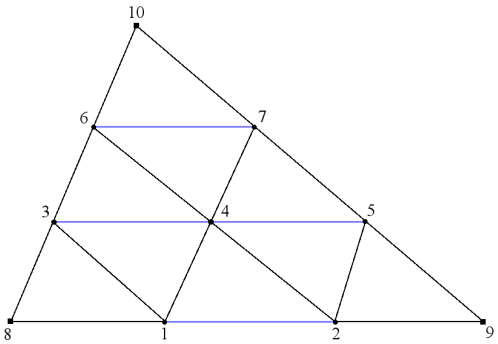

FIGURE 5. Couplings corresponding to the largest entries in the macroelement pivot block (blue) for 3-refinement.

We start with the computation and analysis of the additive preconditioner $C_{E:11}^M$ for the pivot block of the macro element mass matrix $M_{E:11}$.

5.1. **Additive preconditioning for the pivot block of the mass matrix.** Let us consider the element mass matrix (19) and let $\alpha = a/c$ and $\beta = b/c$ with $|a| \leq b \leq c$, where $(\alpha, \beta) \in D$ as illustrated in Fig. 2. Then the pivot block of the macro element mass matrix corresponding to the node numbering presented in Fig. 4 is given by

$$M_{E:11} = \frac{h^2 c \, (\beta + 1)}{24} \begin{pmatrix} 6 & 1 & 2 & 2 & 0 & 0 & 0 \\ 1 & 6 & 0 & 2 & 2 & 0 & 0 \\ 2 & 0 & 6 & 2 & 0 & 1 & 0 \\ 2 & 2 & 2 & 12 & 2 & 2 & 2 \\ 0 & 2 & 0 & 2 & 6 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 6 & 2 \\ 0 & 0 & 0 & 2 & 1 & 2 & 6 \end{pmatrix}.$$

We choose the following additive preconditioner by preserving only the largest (in magnitude) off-diagonal entries as illustrated in Fig. 5:

$$C^M_{E:11} = \frac{h^2 c\,(\beta+1)}{24} \begin{pmatrix} 6 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 12 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 6 \end{pmatrix}. \tag{34}$$

Hence, the additive preconditioner of M_{11} is defined via the macroelement-by-macroelement assembling by

$$C^M_{11} = \sum_{E\in\mathcal{T}^{(k-1)}} R_E^T\, C^M_{E:11}\, R_E. \tag{35}$$

Theorem 2. *The additive preconditioner of M_{11} (35) with (34) yields a relative condition number uniformly bounded by*

$$\kappa((C^M_{11})^{-1}M_{11}) \le \frac{1+\sqrt{\frac{205}{1792}+\frac{3\sqrt{3873}}{1792}}}{1-\sqrt{\frac{205}{1792}+\frac{3\sqrt{3873}}{1792}}} \approx 2.75607, \tag{36}$$

which holds independent of the shape, the size of each element and of the coefficients of the FEM problem.

Proof. In order to obtain the relative condition number of the preconditioned system $\kappa((C^M_{11})^{-1}M_{11})$, we have to solve the local eigenproblem $M_{E:11}\boldsymbol{v_E} = \lambda_E\, C^M_{E:11}\boldsymbol{v_E}$. In the characteristic equation $\det(M_{E:11}-\lambda_E C^M_{E:11}) = 0$ we substitute $\lambda_E = 1-\mu_E$ and obtain the equation

$$(\beta+1)c\,h\,\mu_E\left(40\,\mu_E^2-7\right)\left(896\,\mu_E^4-205\,\mu_E^2+2\right)=0.$$

Since $c\neq 0$, $h\neq 0$ and $\beta\neq -1$, we obtain the following solutions of this equation:

$$\begin{aligned} \mu_E^{(1)} &= 0, \\ \mu_E^{(2/3)} &= \pm\frac{1}{2}\sqrt{\frac{7}{10}} \approx \pm 0.41833, \\ \mu_E^{(4/5)} &= \pm\frac{1}{16}\sqrt{\frac{1}{7}\left(205-3\sqrt{3873}\right)} \approx \pm 0.101054, \\ \mu_E^{(6/7)} &= \pm\sqrt{\frac{205}{1792}+\frac{3\sqrt{3873}}{1792}} \approx \pm 0.467528. \end{aligned}$$

Hence, the largest and smallest eigenvalues $\lambda_E^{\max}$ and $\lambda_E^{\min}$ are given by

$$\lambda_E^{\max} = 1+\sqrt{\frac{205}{1792}+\frac{3\sqrt{3873}}{1792}} \approx 1.46753$$

and

$$\lambda_E^{\min} = 1-\sqrt{\frac{205}{1792}+\frac{3\sqrt{3873}}{1792}} \approx 0.532472,$$

and thus it follows that the local eigenvalue estimate

$$1-\sqrt{\frac{205}{1792}+\frac{3\sqrt{3873}}{1792}} < \lambda_E < 1+\sqrt{\frac{205}{1792}+\frac{3\sqrt{3873}}{1792}}. \tag{37}$$

This estimate leads together with the macroelement-by-macroelement assembling procedure (35) to the relative condition number estimate (36). □

In the next subsection, we compute and analyze the additive preconditioner for the pivot block of the stiffness matrix.

5.2. **Additive preconditioning for the pivot block of the stiffness matrix.** Let us consider now the element stiffness matrix (18) and let again $\alpha = a/c$ and $\beta = b/c$ with $|a| \le b \le c$, where $(\alpha, \beta) \in D$ as illustrated in Fig. 2. Then the pivot block of the macro element stiffness matrix corresponding to the node numbering presented in Fig. 4 is given by

$$K_{E:11} = c \begin{pmatrix} \sigma & -\frac{1}{2} & -\alpha & -\beta & 0 & 0 & 0 \\ -\frac{1}{2} & \sigma & 0 & -\alpha & -\beta & 0 & 0 \\ -\alpha & 0 & \sigma & -1 & 0 & -\frac{\beta}{2} & 0 \\ -\beta & -\alpha & -1 & 2\,\sigma & -1 & -\alpha & -\beta \\ 0 & -\beta & 0 & -1 & \sigma & 0 & -\frac{\alpha}{2} \\ 0 & 0 & -\frac{\beta}{2} & -\alpha & 0 & \sigma & -1 \\ 0 & 0 & 0 & -\beta & -\frac{\alpha}{2} & -1 & \sigma \end{pmatrix},$$

where $\sigma = \alpha + \beta + 1$. We define the additive preconditioner of K_{11} by

$$C_{11}^K = \sum_{E \in \mathcal{T}^{(k-1)}} R_E^T \, C_{E:11}^K \, R_E \tag{38}$$

with

$$C_{E:11}^K = c \begin{pmatrix} \sigma & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2\sigma & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & \sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & \sigma \end{pmatrix}, \tag{39}$$

where we have preserved only the largest (in magnitude) off-diagonal entries of the macroelement stiffness matrix pivot block $K_{E:11}$ in order to get $C_{E:11}^K$ as illustrated in Fig. 5. This preconditioner has the same nonzero pattern as the one for the pivot block of the mass matrix.

Theorem 3. *The additive preconditioner of K_{11} (38) with (39) yields a relative condition number uniformly bounded by*

$$\kappa((C_{11}^K)^{-1} K_{11}) \le \frac{1 + \sqrt{\frac{223}{640} + \frac{3\sqrt{5241}}{640}}}{1 - \sqrt{\frac{223}{640} + \frac{3\sqrt{5241}}{640}}} \approx 10.7185. \tag{40}$$

which holds independent of the shape, the size of each element and of the coefficients of the FEM problem.

Proof. In order to estimate the condition number of the preconditioned pivot block K_{11} we consider the local generalized eigenproblem

$$K_{E:11} \, \mathbf{v}_{E:1} = \lambda_E \, C_{E:11}^K \, \mathbf{v}_{E:1}. \tag{41}$$

We rewrite (41) in the form

$$\mathbf{v}_{E:1}^T \left(K_{E:11} - \lambda_E \, C_{E:11}^K \right) \mathbf{v}_{E:1} = 0,$$

substitute $\lambda_E = 1 - \mu_E$ and define

$$c \, P(\mu_E, \alpha, \beta) = K_{E:11} - (1 - \mu_E) \, C_{E:11}^K$$

with

$$P(\mu_E, \alpha, \beta) := P_0(\mu_E) + \alpha \, P_\alpha(\mu_E) + \beta \, P_\beta(\mu_E),$$

where

$$P_0(\mu_E) := \mu_E \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

$$P_\alpha(\mu_E) := \begin{pmatrix} \mu_E & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & \mu_E & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & \mu_E & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2\,\mu_E & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \mu_E & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -1 & 0 & \mu_E & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & \mu_E \end{pmatrix}$$

and

$$P_\beta(\mu_E) := \begin{pmatrix} \mu_E & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & \mu_E & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \mu_E & 0 & 0 & -\frac{1}{2} & 0 \\ -1 & 0 & 0 & 2\,\mu_E & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & \mu_E & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & \mu_E & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & \mu_E \end{pmatrix}$$

do not depend on α and β. Since $P(\mu_E, \alpha, \beta)$ depends linearly on α and β, this matrix valued function can be only maximal or minimal on the boundary of the domain D (20) illustrated in Fig. 2, hence, either for $-\frac{1}{2} < \alpha \leq 0$ and $\beta = -\frac{\alpha}{\alpha+1}$ or for $\alpha = \beta = 1$. It remains to determine the corresponding μ_E. Let us firstly consider the simplier case $\alpha = \beta = 1$.

In the case $\alpha = \beta = 1$, we solve the characteristic equation corresponding to problem (41) which yields the equation

$$\mu_E \left(107520\mu_E^6 - 87408\mu_E^4 + 9369\mu_E^2 - 78\right) = 0$$

and has the following solutions:

$$\begin{aligned} \mu_E^{(1)} &= 0, \\ \mu_E^{(2/3)} &= \pm\frac{1}{4}\sqrt{\frac{13}{7}} \approx \pm 0.340693, \\ \mu_E^{(4/5)} &= \pm\frac{1}{8}\sqrt{\frac{1}{10}\left(223 - 3\sqrt{5241}\right)} \approx \pm 0.0953263, \\ \mu_E^{(6/7)} &= \pm\sqrt{\frac{223}{640} + \frac{3\sqrt{5241}}{640}} \approx \pm 0.82933. \end{aligned}$$

Hence, the local largest and smallest eigenvalues corresponding to the case $\alpha = \beta = 1$ are given by

$$\lambda_E^{\max} = 1 + \sqrt{\frac{223}{640} + \frac{3\sqrt{5241}}{640}} \approx 1.82933 \text{ and}$$

$$\lambda_E^{\min} = 1 - \sqrt{\frac{223}{640} + \frac{3\sqrt{5241}}{640}} \approx 0.17067, \text{ respecively.}$$

Now, we consider the second case, i.e., $-\frac{1}{2} < \alpha \leq 0$ and $\beta = -\frac{\alpha}{\alpha+1}$. The characteristic equation corresponding to (41) together with $c \neq 0$ and with the substitution $\lambda_E = 1 - \mu_E$ is given by

$$\begin{aligned}
&\frac{\alpha^4\,\mu_E}{32(\alpha+1)^7}[16\left(\alpha^2+\alpha+1\right)\left(2\alpha^2+\alpha+1\right)(\alpha(\alpha+2)+2)^2(\alpha(2\alpha+3)+3)\,\mu_E^6\\
&-2\alpha^2(\alpha(\alpha(\alpha(\alpha(\alpha(8\alpha(\alpha+7)+191)+421)+589)+457)+157)+4)+1)\\
&-6(\alpha(\alpha(\alpha(\alpha(\alpha(2\alpha(\alpha(12\alpha(\alpha+7)+293)+647)+2001)+2229)+1807)\\
&+1026)+399)+95)+19)\,\mu_E^4+3(\alpha(\alpha(\alpha(\alpha(\alpha(2\alpha(\alpha(16\alpha(\alpha+7)+375)\\
&+781)+2203)+2119)+1397)+598)+157)+5)+1)\,\mu_E^2]=0
\end{aligned}$$

for $-\frac{1}{2} < \alpha \leq 0$. The first solution is $\mu_E^{(1)} = 0$ and $\lambda_E^{(1)} = 1$. Moreover, the equation is fulfilled for $\alpha = 0$. So we consider the case $\alpha \in (-\frac{1}{2}, 0)$. We substitute now $\nu_E = \mu_E^2$ and solve the equation by Cardano's formula. Our equation has the form

$$A(\alpha)\,\nu_E^3 + B(\alpha)\,\nu_E^2 + C(\alpha)\,\nu_E + D(\alpha) = 0 \tag{42}$$

with, e.g., $A(\alpha) = 16\left(\alpha^2+\alpha+1\right)\left(2\alpha^2+\alpha+1\right)(\alpha(\alpha+2)+2)^2(\alpha(2\alpha+3)+3)$. We divide by $A(\alpha) \neq 0$ and obtain an equation of the form

$$\nu_E^3 + a(\alpha)\,\nu_E^2 + b(\alpha)\,\nu_E + c(\alpha) = 0,$$

where, e.g.,

$$\begin{aligned}
a(\alpha) &= B(\alpha)/A(\alpha)\\
&= \frac{3}{8}\left(-\frac{6(\alpha+1)}{\alpha^2+\alpha+1} + \frac{6(\alpha+1)}{2\alpha^2+\alpha+1} + \frac{-10\alpha-7}{\alpha(\alpha+2)+2}\right.\\
&\left.+\frac{26(\alpha+1)}{\alpha(2\alpha+3)+3} - \frac{3}{(\alpha(\alpha+2)+2)^2} - 6\right).
\end{aligned}$$

Since our matrices are symmetric, we have only real eigenvalues. So Cardano's formula predicts three different real solutions of the cubic equation (42). Using the substitution $\nu_E = z - \frac{a(\alpha)}{3}$, we obtain the equation

$$z^3 + p(\alpha)\,z + q(\alpha) = 0,$$

where

$$\begin{aligned}
p(\alpha) &= b(\alpha) - \frac{a(\alpha)^2}{3},\\
q(\alpha) &= \frac{2a(\alpha)^3}{27} - \frac{a(\alpha)b(\alpha)}{3} + c(\alpha),
\end{aligned}$$

and the three real solutions are computed by

$$\begin{aligned}
z^{(1)} &= \sqrt{-\frac{4}{3}p(\alpha)}\cos\left(\frac{1}{3}\arccos\left(-\frac{q(\alpha)}{2}\sqrt{-\frac{27}{p(\alpha)^3}}\right)\right),\\
z^{(2)} &= -\sqrt{-\frac{4}{3}p(\alpha)}\cos\left(\frac{1}{3}\arccos\left(-\frac{q(\alpha)}{2}\sqrt{-\frac{27}{p(\alpha)^3}}\right)+\frac{\pi}{3}\right),\\
z^{(3)} &= -\sqrt{-\frac{4}{3}p(\alpha)}\cos\left(\frac{1}{3}\arccos\left(-\frac{q(\alpha)}{2}\sqrt{-\frac{27}{p(\alpha)^3}}\right)-\frac{\pi}{3}\right).
\end{aligned}$$

The solutions $\nu_E = \mu_E^2$ are given by

$$\nu_E^{(i)}(\alpha) = z^{(i)} - \frac{a(\alpha)}{3} \qquad \text{with } i = 1, 2, 3.$$

The three solutions - in dependence of α - are illustrated in Fig. 6. We see that $\nu^{(1)}(\alpha)$ corresponds to the largest value for μ_E and has its maximum for $\alpha = -1/2$.

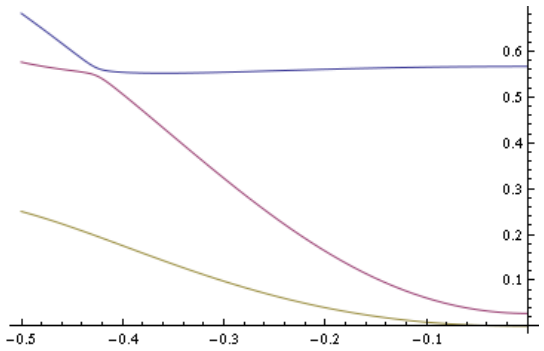

FIGURE 6. The three solutions $\nu^{(1)}(\alpha)$, $\nu^{(2)}(\alpha)$ and $\nu^{(3)}(\alpha)$ for $\alpha \in (-1/2, 0)$.

This can be proven by a standard tool from symbolic computation called Cylindrical Algebraic Decomposition (CAD), see [10, 11, 12]. We used the Mathematica built-in commands "CylindricalDecomposition" and "Resolve" for the proof. The computation takes less than 0.1 seconds. For $\alpha = -1/2$,

$$\nu^{(1)}(-1/2) = \frac{1}{800}\left(503 + 3\sqrt{201}\right) \approx 0.681915$$

and the corresponding maximal and minimal eigenvalue are given by $\lambda_E^{\max} = 1 + \sqrt{\nu^{(1)}(-1/2)} \approx 1.82578$ and $\lambda_E^{\min} = 1 - \sqrt{\nu^{(1)}(-1/2)} \approx 0.174218$, respecively. The outcome of comparing these eigenvalues to the ones for the case $\alpha = \beta = 1$ is that the maximal and minimal eigenvalues are both attained for $\alpha = \beta = 1$ which leads to the following local eigenvalue estimate:

$$0.17067 \approx 1 - r \;<\; \lambda_E \;<\; 1 + r \approx 1.82933, \tag{43}$$

where $r = \sqrt{\frac{223}{640} + \frac{3\sqrt{5241}}{640}}$. Together with the macroelement-by-macroelement assembling procedure (38), we finally arrive at the condition number estimate

$$\begin{aligned}\kappa((C_{11}^K)^{-1}K_{11}) &\leq \frac{1 + \sqrt{\frac{223}{640} + \frac{3\sqrt{5241}}{640}}}{1 - \sqrt{\frac{223}{640} + \frac{3\sqrt{5241}}{640}}} \\ &= \frac{1}{33}\left(106 + \sqrt{5241} + 2\sqrt{3847 + 53\sqrt{5241}}\right) \approx 10.7185.\end{aligned}$$

□

Remark 3. *Instead of using CAD for proving that the maximum and minimum of $\nu^{(i)}(\alpha)$ with $i = 1, 2, 3$ is attained for $\nu^{(1)}(-1/2)$, we can, alternatively, prove a less sharper bound by hand if we estimate the cosine in all formulas of $z^{(i)}$, $i = 1, 2, 3$, by one. This yields the following bound for all three solutions:*

$$\nu_E^{(i)}(\alpha) \leq \sqrt{-\frac{4}{3}p(\alpha)} \cdot 1 - \frac{a(\alpha)}{3} =: \nu_{\max}(\alpha).$$

This bound can be written as

$$\nu_{\max}(\alpha) = -\frac{a(\alpha)}{3} + \sqrt{-\frac{4}{3}p(\alpha)} = -\frac{1}{8}f(\alpha) + \frac{1}{4}\sqrt{f(\alpha)^2 - 4\,g(\alpha)}$$

where $f(\alpha) = \frac{8}{3}a(\alpha)$ and $g(\alpha) = \frac{16}{3}b(\alpha)$. As we see in Fig. 7, $\nu_{\max}(\alpha)$ has its maximum at the boundary for $\alpha = -1/2$. Now, we want to prove this statement. Since f and g are rational functions, we want to determine some non-rational functions which are upper and lower bounds for them and which are equal for $\alpha = -1/2$ and

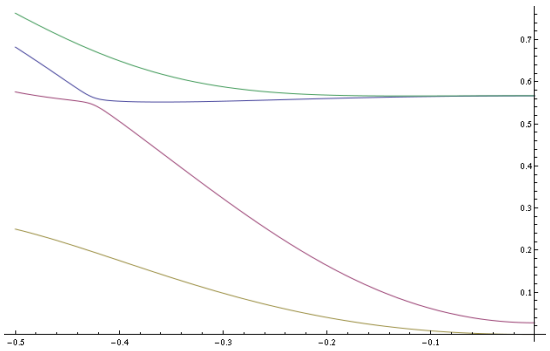

FIGURE 7. $\nu^{(1)}(\alpha)$, $\nu^{(2)}(\alpha)$, $\nu^{(3)}(\alpha)$ and their bound $\nu_{\max}(\alpha)$ for $\alpha \in (-1/2, 0)$.

$\alpha = 0$. *More precisely, we want to determine some functions* $\overline{f}(\alpha)$ *and* $\underline{g}(\alpha)$ *such that we can bound* $\nu_{\max}$ *from above, i.e.,*

$$\begin{aligned}\nu_{\max}(\alpha) &= -\frac{1}{8}f(\alpha) + \frac{1}{4}\sqrt{f(\alpha)^2 - 4\,g(\alpha)} \\ &\le \frac{1}{8}\overline{f}(\alpha) + \frac{1}{4}\sqrt{\overline{f}(\alpha)^2 - 4\,\underline{g}(\alpha)} =: \overline{\nu_{\max}}(\alpha).\end{aligned}$$

The idea is to choose quadratic functions as upper bound for $-f$ *and as lower bound for* g*. We compute three points of* $-f$ *and* g *and then fit quadratic polynomials through these points and we obtain, e.g., the following two quadratic functions for* $-f$ *and* g*:*

$$\overline{f}(\alpha) = \frac{119}{27}\alpha^2 - \frac{1802}{675}\alpha + \frac{19}{12}, \qquad \underline{g}(\alpha) = \frac{232}{15}\alpha^2 + \frac{9}{25}\alpha + \frac{1}{12}.$$

Together with $\overline{f}$ *and* $\underline{g}$*, the function* $\overline{\nu_{\max}}(\alpha)$ *does not have a local extremum and hence its maximum is attained for* $\alpha = -1/2$*. Here,* $\overline{\nu_{\max}}(-1/2) = \nu_{\max}(-1/2) = \frac{201}{400} + \frac{\sqrt{2701}}{200} \approx 0.762356$*. In this case,* $P(\mu_E, \alpha, \beta)$ *is maximal and minimal for* $\alpha = -1/2$ *and* $\beta = 1$*. Together with the macroelement-by-macroelement assembling (38), we obtain the following condition number estimate:*

$$\kappa((C_{11}^K)^{-1}K_{11}) \le \frac{1 + \sqrt{\overline{\nu_{\max}}(-1/2)}}{1 - \sqrt{\overline{\nu_{\max}}(-1/2)}} \approx 14.7641.$$

5.3. **Additive preconditioning for the pivot block of the whole system.**

Theorem 4. *The additive preconditioner (33) of* A_{11} *with the additive preconditioners for the pivot block of the mass matrix (35) and for the stiffness matrix (38), where we have the preconditioners (34) and (39) corresponding to the macroelements, yield altogether a relative condition number uniformly bounded by*

$$\kappa((C_{11})^{-1}A_{11}) \le \frac{1 + \sqrt{\frac{223}{640} + \frac{3\sqrt{5241}}{640}}}{1 - \sqrt{\frac{223}{640} + \frac{3\sqrt{5241}}{640}}} \approx 10.7185, \tag{44}$$

which holds independent of the shape, the size of each element and of the coefficients of the FEM problem.

Proof. It follows from the macroelement-by-macroelement assembling that

$$\begin{aligned}\mathbf{v}_1^T A_{11}\mathbf{v}_1 &= \sum_{E\in\mathcal{T}^{(k-1)}} \mathbf{v}_{E:1}^T\, R_E^T\, A_{E:11}\, R_E\, \mathbf{v}_{E:1} \\ &= \sum_{E\in\mathcal{T}^{(k-1)}} \mathbf{v}_{E:1}^T\, R_E^T\, (K_{E:11} + \tilde{\mu}_E\, M_{E:11})\, R_E\, \mathbf{v}_{E:1}.\end{aligned}$$

Using the local eigenvalue estimates (37) and (43) comming from the local eigenproblem for $M_{E:11}$ and $C^M_{E:11}$ and from the local generalized eigenproblem for $K_{E:11}$ and $C^K_{E:11}$, respecively, yields the following upper bound:

$$\mathbf{v}_1^T A_{11}\mathbf{v}_1 = \sum_{E\in\mathcal{T}^{(k-1)}} \mathbf{v}_{E:1}^T R_E^T K_{E:11} R_E \mathbf{v}_{E:1} + \sum_{E\in\mathcal{T}^{(k-1)}} \mathbf{v}_{E:1}^T R_E^T \tilde{\mu}_E M_{E:11} R_E \mathbf{v}_{E:1}$$
$$\leq \sum_{E\in\mathcal{T}^{(k-1)}} \lambda^{\max}_{K,E} \mathbf{v}_{E:1}^T R_E^T C^K_{E:11} R_E \mathbf{v}_{E:1} + \sum_{E\in\mathcal{T}^{(k-1)}} \lambda^{\max}_{M,E} \mathbf{v}_{E:1}^T R_E^T \tilde{\mu}_E C^M_{E:11} R_E \mathbf{v}_{E:1}.$$

Taking the maximum over the two eigenvalues $\lambda^{\max}_{K,E}$ and $\lambda^{\max}_{M,E}$, i.e.,

$$\lambda^{\max}_{A,E} = \max\{\lambda^{\max}_{K,E}, \lambda^{\max}_{M,E}\},$$

leads to

$$\begin{aligned}
\mathbf{v}_1^T A_{11}\mathbf{v}_1 &= \sum_{E\in\mathcal{T}^{(k-1)}} \mathbf{v}_{E:1}^T R_E^T A_{E:11} R_E \mathbf{v}_{E:1} \\
&\leq \sum_{E\in\mathcal{T}^{(k-1)}} \lambda^{\max}_{A,E} \mathbf{v}_{E:1}^T R_E^T \left(C^K_{E:11} + \tilde{\mu}_E C^M_{E:11}\right) R_E \mathbf{v}_{E:1} \\
&\leq \lambda^{\max}_A \sum_{E\in\mathcal{T}^{(k-1)}} \mathbf{v}_{E:1}^T R_E^T \left(C^K_{E:11} + \tilde{\mu}_E C^M_{E:11}\right) R_E \mathbf{v}_{E:1} \\
&= \lambda^{\max}_A \sum_{E\in\mathcal{T}^{(k-1)}} \mathbf{v}_{E:1}^T R_E^T C_{E:11} R_E \mathbf{v}_{E:1} = \lambda^{\max}_A \mathbf{v}_1^T C_{11} \mathbf{v}_1,
\end{aligned}$$

where $\lambda^{\max}_A = 1 + \sqrt{\frac{223}{640} + \frac{3\sqrt{5241}}{640}}$. Similarly, one proves

$$\mathbf{v}_1^T A_{11}\mathbf{v}_1 > \lambda^{\min}_A \mathbf{v}_1^T C_{11} \mathbf{v}_1$$

with $\lambda^{\min}_A = 1 - \sqrt{\frac{223}{640} + \frac{3\sqrt{5241}}{640}}$. Altogether, both inequalities finally yield the condition number estimate (44). □

6. Stabilization polynomials of higher degree

As briefly discussed in Section 3, we combine hierarchical basis preconditioners with stabilization techniques in order to obtain an AMLI method of optimal order. In this section, we discuss the construction of the matrix polynomials which occur in the approximations of the Schur complements at levels $k = 1, ..., K$, not only of degree 1 and 2 but also of higher degree since we consider the 3-refinement and so the degree of the polynomial has to be chosen higher in order to fulfill the optimality conditions (16) or (17). Efficient choices for matrix polynomials $P^{(k)}(t) = P_{\upsilon_k}(t)$ approximating the Schur complements are based on Chebyshev polynomials, see [5, 6] as well as [1, 18]. The matrix polynomials for the interval $[\alpha, 1]$ with $0 < \alpha < 1$ are chosen by

$$P_{\upsilon_k}(t) = \frac{T_{\upsilon_k}\left(\frac{1+\alpha-2\,t}{1-\alpha}\right) + 1}{T_{\upsilon_k}\left(\frac{1+\alpha}{1-\alpha}\right) + 1}, \tag{45}$$

where $T_{\upsilon_k}(t)$ is the Chebyshev polynomial of the first kind defined via the recursion

$$\begin{aligned}
&T_{\upsilon_k}(t) = 2\,t\,T_{\upsilon_k-1}(t) - T_{\upsilon_k-2}(t), \quad \upsilon_k = 2, 3, ..., \\
&T_0(t) = 1, \quad T_1(t) = t.
\end{aligned}$$

Next, we define the polynomial $Q^{(k)}(t) = Q_{\upsilon_k-1}(t)$ by

$$Q_{\upsilon_k-1}(t) = \frac{1 - P_{\upsilon_k}(t)}{t} = q_0^{(k)} + q_1^{(k)} t + ... + q^{(k)}_{\upsilon_k-1} t^{\upsilon_k-1}. \tag{46}$$

In Table 1, Table 2, Table 3 and Table 4 we illustrate the coefficients $q_i^{(k)}$, $i \in \{0, 1, ..., 4\}$, of all polynomials $Q_{\upsilon_k-1}(t)$ for $\upsilon_k \in \{1, 2, ..., 5\}$.

TABLE 1. Coefficients of the polynomials $Q_0(t)$ and $Q_1(t)$

$q_i^{(k)}$	$\upsilon_k = 1$
$i = 0$	$q_0^{(1)} = 1$

$q_i^{(k)}$	$\upsilon_k = 2$
$i = 0$	$q_0^{(2)} = \frac{4}{1+\alpha}$
$i = 1$	$q_1^{(2)} = -\frac{4}{(1+\alpha^2)}$

TABLE 2. Coefficients of the polynomial $Q_2(t)$

$q_i^{(k)}$	$\upsilon_k = 3$
$i = 0$	$q_0^{(3)} = \frac{9+3\alpha}{1+3\alpha}$
$i = 1$	$q_1^{(3)} = -\frac{24(1+\alpha)}{(1+3\alpha)^2}$
$i = 2$	$q_2^{(3)} = \frac{16}{(1+3\alpha)^2}$

TABLE 3. Coefficients of the polynomial $Q_3(t)$

$q_i^{(k)}$	$\upsilon_k = 4$
$i = 0$	$q_0^{(4)} = \frac{16(1+\alpha)}{1+\alpha(6+\alpha)}$
$i = 1$	$q_1^{(4)} = -\frac{16(5+\alpha(14+5\alpha))}{(1+\alpha(6+\alpha)^2}$
$i = 2$	$q_2^{(4)} = \frac{128(1+\alpha)}{(1+\alpha(6+\alpha)^2}$
$i = 3$	$q_3^{(4)} = -\frac{64}{(1+\alpha(6+\alpha)^2}$

TABLE 4. Coefficients of the polynomial $Q_4(t)$

$q_i^{(k)}$	$\upsilon_k = 4$
$i = 0$	$q_0^{(5)} = \frac{5(5+\alpha(10+\alpha))}{1+5\alpha(2+\alpha)}$
$i = 1$	$q_1^{(5)} = -\frac{40(1+\alpha)(5+\alpha(22+5\alpha))}{(1+5\alpha(2+\alpha))^2}$
$i = 2$	$q_2^{(5)} = \frac{80(7+\alpha(18+7\alpha))}{(1+5\alpha(2+\alpha))^2}$
$i = 3$	$q_3^{(5)} = -\frac{640(1+\alpha)}{(1+5\alpha(2+\alpha))^2}$
$i = 4$	$q_4^{(5)} = \frac{256}{(1+5\alpha(2+\alpha))^2}$

Using a 3-refinement, we have to choose a stabilization polynomial (45) of order 5, which corresponds to a polynomial (46) of order 4, e.g., the polynomial $Q_4(t)$ in order to obtain a method of optimal order of computational complexity.

7. A PRACTICAL EXAMPLE

Let the domain $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain with the boundary $\Gamma := \partial\Omega$ and let denote the space-time cylinder by $Q_T := \Omega \times (0,T)$ and its mantle boundary by $\Sigma_T := \Gamma \times (0,T)$. We consider the following time-periodic parabolic distributed optimal control problem:

$$\min_{y,u} \mathcal{J}(y,u) := \frac{1}{2}\int_0^T \int_\Omega [y(\boldsymbol{x},t) - y_d(\boldsymbol{x},t)]^2 \, d\boldsymbol{x}\, dt + \frac{\lambda}{2}\int_0^T \int_\Omega [u(\boldsymbol{x},t)]^2 \, d\boldsymbol{x}\, dt$$

subject to the time-periodic parabolic PDE constraints

$$\begin{aligned} \sigma(\boldsymbol{x})\frac{\partial}{\partial t}y(\boldsymbol{x},t) - \nabla\cdot(\nu(\boldsymbol{x})\nabla y(\boldsymbol{x},t)) &= u(\boldsymbol{x},t), \quad \text{in } Q_T,\\ y(\boldsymbol{x},t) &= 0, \quad \text{on } \Sigma_T,\\ y(\boldsymbol{x},0) &= y(\boldsymbol{x},T), \quad \text{in } \Omega, \end{aligned}$$

with strictly positive and uniformly bounded coefficients σ and ν, i.e., $0 < \underline{\sigma} \leq \sigma(\boldsymbol{x}) \leq \overline{\sigma}$ and $0 < \underline{\nu} \leq \nu(\boldsymbol{x}) \leq \overline{\nu}$, for $\boldsymbol{x} \in \Omega$ and which are piecewise constant. The state is denoted by y and the desired state by y_d which we try to reach via a suitable control u. The parameter $\lambda > 0$ provides a weighting of the cost of the control in the cost functional $\mathcal{J}(\cdot,\cdot)$. We formulate the optimality system und discretize the problem by the multiharmonic finite element method, see [14, 19, 20]. More precisely, we expand all functions into Fourier series in time, truncate them at an index N, e.g.,

$$y_d(\boldsymbol{x},t) \approx \sum_{k=0}^{N}[y_{dk}^c(\boldsymbol{x})\cos(k\omega t) + y_{dk}^s(\boldsymbol{x})\sin(k\omega t)],$$

where its Fourier coefficients are given by

$$y_{dk}^c(\boldsymbol{x}) = \frac{2}{T}\int_0^T y_d(\boldsymbol{x},t)\cos(k\omega t)\,dt, \qquad y_{dk}^s(\boldsymbol{x}) = \frac{2}{T}\int_0^T y_d(\boldsymbol{x},t)\sin(k\omega t)\,dt,$$

and then approximate the Fourier coefficients from $V = H_0^1(\Omega)$ by finite element functions from a space $V_h \subset V$ as presented in (3). Finally, we have to solve the following linear saddle point system:

$$\underbrace{\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}}_{=:\mathcal{A}} \begin{pmatrix} \underline{y} \\ \underline{p} \end{pmatrix} = \underbrace{\begin{pmatrix} \underline{f} \\ 0 \end{pmatrix}}_{=:f}, \tag{47}$$

where, for alle modes $k = 1, ..., N$, we have that

$$A := \begin{pmatrix} M_h & 0 \\ 0 & M_h \end{pmatrix},\ B := \begin{pmatrix} -K_h & -k\omega M_{\sigma,h} \\ k\omega M_{\sigma,h} & -K_h \end{pmatrix},$$

$C := \lambda^{-1}A$ and

$$\underline{f} := \begin{pmatrix} \underline{y}_{dk}^c \\ \underline{y}_{dk}^s \end{pmatrix},\ \underline{y} := \begin{pmatrix} \underline{y}_k^c \\ \underline{y}_k^s \end{pmatrix},\ \underline{p} := \begin{pmatrix} \underline{p}_k^c \\ \underline{p}_k^s \end{pmatrix}.$$

The matrices M_h, $M_{h,\sigma}$ and K_h correspond to the mass matrix, the weighted mass matrix and the stiffness matrix, respectively. The entries of the weighted mass and stiffness matrix are computed by

$$M_{\sigma,h}^{ij} = \int_\Omega \sigma\,\phi_K^{(i)}\phi_K^{(j)}\,d\boldsymbol{x}, \quad K_h^{ij} = \int_\Omega \nu\,\nabla\phi_K^{(i)}\cdot\nabla\phi_K^{(j)}\,d\boldsymbol{x},$$

with $i, j = 1, ..., N^{(K)} = N_h$, whereas the nodal parameter vectors are given by, e.g.,

$$\underline{y}_{dk}^c = \Big[\int_\Omega y_{dk}^c\phi_K^{(j)}\,d\boldsymbol{x}\Big]_{j=1,\ldots,N_h}.$$

Saddle point problems of the form (47) can be solved by a preconditioned minimal residual (MINRES) method, see [25]. In [14], the authors presented a preconditioner for system (47) and proved that this preconditioner yields a robust and fast convergence rate. This preconditioner is given by

$$\mathcal{P} = \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & \lambda^{-1}D & 0 \\ 0 & 0 & 0 & \lambda^{-1}D \end{pmatrix}$$

with the matrices $D := \sqrt{\lambda}\, K_h + k\omega\sqrt{\lambda}\, M_{\sigma,h} + M_h$ implying the condition number estimate $\kappa_{\mathcal{P}}(\mathcal{P}^{-1}\mathcal{A}) := \|\mathcal{P}^{-1}\mathcal{A}\|_{\mathcal{P}}\, \|\mathcal{A}^{-1}\mathcal{P}\|_{\mathcal{P}} \leq \sqrt{3}$ for all modes $k = 1, ..., N$. All parameters are at least constant on the coarsest mesh partioning and hence we could use the AMLI preconditioner presented in this paper to implement the preconditioner D practically leading altogether to a robust and optimal AMLI preconditioned MINRES method.

8. Conclusions

The main contribution of this paper is the verification of the optimality conditions for linear AMLI preconditioners constructed in the framework of hierarchical splittings of lowest-order conforming finite element space for reaction-diffusion type problems. A new estimate of the constant γ in the strengthened Cauchy-Bunyakowski-Schwarz inequality has been presented for the mass matrix in case of a general m-refinement. Moreover, an additive preconditioner for the pivot blocks arising in the recursive two-by-two block factorization has been analyzed for the case $m = 3$. The derived uniform condition number estimates together with the verification of the optimality conditions guarantee the existence of optimal linear AMLI methods for linear systems with weighted sums of stiffness and mass matrices.

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References

[1] O. Axelsson. *Iterative Solution Methods*. Cambridge University Press, 1994.

[2] O. Axelsson and V. Barker. *Finite element solution of boundary value problems: theory and computations*. Academic Press, Orlando, 1984.

[3] O. Axelsson and R. Blaheta. Two simple derivations of universal bounds for the C.B.S. inequality constant. *Appl. Math.*, 49:57–72, 2004.

[4] O. Axelsson and A. Padiy. On the additive version of the algebraic multilevel iteration method for anisotropic elliptic problems. *SIAM J. Sci. Comput.*, 20(5):1807–1830, 1999.

[5] O. Axelsson and P. Vassilevski. Algebraic multilevel preconditioning methods I. *Numer. Math.*, 56:157–177, 1989.

[6] O. Axelsson and P. Vassilevski. Algebraic multilevel preconditioning methods II. *SIAM J. Numer. Anal.*, 27:1569–1590, 1990.

[7] O. Axelsson and P. Vassilevski. Variable-step multilevel preconditioning methods, I: Self-adjoint and positive definite elliptic problems. *Numer. Linear Algebra Appl.*, 1(1):75–101, 1994.

[8] O. Axelsson and P. Vassilevski. *The AMLI Method: an Algebraic Multilevel Iteration Method for Positive Definite Sparse Matrices*. University of Nijmegen, Departement of Mathematics, 1999.

[9] P. Ciarlet. *The finite element method for elliptic problems*. North-Holland Publishing Company, Amsterdam-New York-Oxford, 1978. Republished by SIAM in April 2002.

[10] G. Collins. Quantifier elimination for real closed fields by cylindrical algebraic decomposition. *Automata theory and formal languages (Second GI Conf., Kaiserslautern, 1975), Lecture Notes in Comput. Sci.*, 33:134–183, 1975.

[11] G. Collins and H. Hong. Partial cylindrical algebraic decomposition for quantifier elimination. *Quantifer elimination and cylindrical algebraic decomposition (Linz, 1993), Texts Monogr. Symbol. Comput.*, pages 174–200, 1998.

[12] M. Kauers. How to use cylindrical algebraic decomposition. *Seminaire Lotharingien de Combinatoire*, 65(B65a):1–16, 2011.

[13] M. Kollmann and M. Kolmbauer. A preconditioned MinRes solver for time-periodic parabolic optimal control problems. *Numer. Linear Algebra Appl.*, 2012.
[14] M. Kollmann, M. Kolmbauer, U. Langer, M. Wolfmayr, and W. Zulehner. A finite element solver for a multiharmonic parabolic optimal control problem. *Comput. Math. Appl.*, 65(3):469–486, 2013.
[15] J. Kraus. An algebraic preconditioning method for M-matrices: Linear versus non-linear multilevel iteration. *Numer. Linear Algebra Appl.*, 9(8):599–618, 2002.
[16] J. Kraus. Additive Schur complement approximation and application to multilevel preconditioning. *SIAM J. Sci. Comput.*, 34(6):A2872–A2895, 2012.
[17] J. Kraus, M. Lymbery, and S. Margenov. Robust multilevel methods for quadratic finite element anisotropic elliptic problems. *Numer. Linear Algebra Appl.*, 2013. (to appear).
[18] J. Kraus and S. Margenov. *Robust Algebraic Multilevel Methods and Algorithms*, volume 5 of *Radon Series on Computational and Applied Mathematics*. de Gruyter, Berlin, 2009.
[19] U. Langer and M. Wolfmayr. Multiharmonic finite element solvers for time-periodic parabolic optimal control problems. *Proc. Appl. Math. Mech. (PAMM)*, 12(1):687–688, 2012.
[20] U. Langer and M. Wolfmayr. Multiharmonic finite element analysis of a time-periodic parabolic optimal control problem. DK-Report 2013-01, Johannes Kepler University, Linz, 2013.
[21] R. Lazarov and S. Margenov. CBS constants for graph-Laplacians and application to multilevel methods for discontinuous Galerkin systems. *Journal of Complexity*, 4-6:498–515, 2007.
[22] M. Lymbery and S. Margenov. Robust semi-coarsening multilevel preconditioning of biquadratic fem systems. *Cent. Eur. J. Math.*, 10(1):357–369, 2012.
[23] J. Maitre and F. Musy. The contraction number of a class of two-level methods: An exact evaluation for some finite element subspaces and model problems. *Lect. Notes Math.*, 960:535–544, 1982.
[24] Y. Notay and P. Vassilevski. Recursive Krylov-based multigrid cycles. *Numer. Linear Algebra Appl.*, 15(5):473–487, 2008.
[25] C. Paige and M. Saunders. Solution of sparse indefinite systems of linear equations. *SIAM J. Numer. Anal.*, 12(4):617–629, 1975.

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