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EXISTENCE AND UNIQUENESS OF EDDY CURRENT PROBLEMS IN BOUNDED AND UNBOUNDED DOMAINS

MICHAEL KOLMBAUER

Abstract. This work is devoted to providing existence and uniqueness results for time-dependent eddy current problems in bounded and unbounded domains. Therein we combine well known results for abstract evolution equations with boundary reduction methods like harmonic extensions and boundary integral operators.

1. Introduction

Eddy current problems are fundamental different for conducting and non-conducting regions. While in conducting regions the problems are of "parabolic" type, in non-conducting regions the problems reduce to "elliptic" ones. In this work we want to analyze these PDEs of mixed type and provide existence and uniqueness results.

In a conducting domain $\Omega_1 \subset \mathbb{R}^3$, the conductivity can be assumed to be piecewise constant and uniformly positive, i.e. $\sigma \geq \sigma_0 > 0$ almost everywhere. Inside the conductor Ω_1, the relation between the magnetic field $\mathbf{H}$ and the magnetic field density $\mathbf{B}$ can be in general nonlinear. Neglecting the effects of hysteresis, the relation is given by the **B-H** curve: $\mathbf{H} = \nu_1(\mathbf{B})\mathbf{B}$, where the reluctivity ν_1 is given by a continuous function $\nu_1 : \mathbb{R}_0^+ \to \mathbb{R}^+$, satisfying the following properties: $s \mapsto \nu_1(s)s$ is strictly monotone and Lipschitz continuous and $s \mapsto \nu_1(s)$ is uniformly positive and bounded. These properties are an immediate consequence of the physical background (see e.g. [10]). We mention, that the reluctivity satisfies the relation $\nu = \mu^{-1}$, where μ is the magnetic permeability. The excitation $\mathbf{f_1}$ is provided by impressed currents or source currents. For simplicity we neglect permanent magnets. By introducing a vector potential $\mathbf{u}$ for the magnetic field density $\mathbf{B} = \mathbf{curl}\,\mathbf{u}$, the equation in the conducting domain has the following parabolic structure:

$$\sigma(x)\frac{\partial \mathbf{u}}{\partial t} + \mathbf{curl}\,(\nu_1(|\mathbf{curl}\,\mathbf{u}|)\,\mathbf{curl}\,\mathbf{u}) = \mathbf{f_1}.$$

In a non-conducting domain Ω_2, the conductivity vanishes, i.e. $\sigma = 0$. Additionally the reluctivity $\nu_2 > 0$ is constant and we assume that there is no source. Hence we deal with a stationary and elliptic **curl-curl** problem:

$$\mathbf{curl}\,(\nu_2\,\mathbf{curl}\,\mathbf{u}) = \mathbf{0}.$$

A non-conducting domain Ω_2 corresponds to the unbounded air region, and therefore is unbounded in general. Nevertheless we also analyze the case of a bounded air region, since in many applications, the unbounded exterior domain is approximated by a bounded one with homogeneous boundary conditions imposed some distance away from the conductor.

In order to provide existence and uniqueness results for a general eddy current problem consisting of both conducting and non-conducting domains, the main tool is the reduction of the full computational domain to the conducting domain only.

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This can be achieved by either using the framework of pde-harmonic extensions or by the framework of boundary integral operators.

Eddy current problems in bounded domains have already been analyzed in [3, 4]. They used pde-harmonic extensions to reduce the full computational domain to the conducting domains only and provided existence and uniqueness results in special gauged spaces. Nevertheless we want to clarify their proving techniques and the computational details. For other works using similar techniques we mention [2].

In order to extend the existence and uniqueness theory also to the case of unbounded domains, in principle the same approach of pde-harmonic extensions can be used. The drawback of the latter mentioned approach is the need for introducing weighted Sobolev spaces, since we are dealing with an unbounded domain. In order to avoid this, we prefer to use the theoretical framework of boundary integral operators. Additionally this approach directly offers a starting point for a domain decomposition method in the terms of a FEM-BEM (Finite Element-Boundary Element) coupling.

Indeed the symmetric coupling of eddy current problems in the frequency domain is well understood [9]. In contrast to the latter mentioned approach, we do not switch from the time domain to the frequency domain, and hence we have to deal with a time-dependent problem. Nevertheless we can combine well known existence and uniqueness results for parabolic problems [12, 13] and the technique of symmetric coupling [8] to obtain existence and uniqueness also in the time domain. For another approach using these techniques for time-dependent eddy current problems we mention [1].

The rest of the paper is organized as follows: In Section 2 we provide standard existence and uniqueness results for abstract evolution equations. After that, the basic function spaces and traces for Maxwell's equations are introduced. Furthermore we collect some useful results for eddy current problems in conducting domains. In Section 3 and 4 the main results, the existence and uniqueness of eddy current problems are presented for the case of unbounded and bounded domains, respectively.

2. Some preliminary results

2.1. **Existence and uniqueness of abstract evolution equations.** The basis for proving existence and uniqueness of degenerated parabolic problems, is the abstract theory for abstract evolution equations. For details we refer to [12] and [13] for linear and nonlinear equations, respectively. We just quote the resulting theorems for operator equations of parabolic type.

Theorem 1 (Nonlinear)**.** *Let $V \subset H \subset V^*$ be an evolution triple. Let $A : V \to V^*$ be a hemicontinuous, monotone, coercive and bounded operator. Suppose furthermore that $F \in L_2((0,T),V^*)$ and $u_0 \in H$ be given. Then the initial value problem*

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}u(t) + A(u(t)) &= F(t), && \text{in } L_2(0,T;V^*) \\ u(0) &= u_0, && \text{in } H \end{aligned}$$

has a unique solution $u \in L_2((0,T),V)$ with weak derivative $\dot{u} \in L_2((0,T),V^*)$.

Proof. [13, Theorem 30.A], see also [13, Corollary 30.12] □

Theorem 2 (Linear)**.** *Let $V \subset H \subset V^*$ be an evolution triple. Let $A : V \to V^*$ be a linear, coercive and bounded operator. Suppose furthermore that $F \in L_2(0,T;V^*)$ and $u_0 \in H$ be given. Then the initial value problem*

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}u(t) + Au(t) &= F(t), && \text{in } L_2(0,T;V^*) \\ u(0) &= u_0, && \text{in } H \end{aligned}$$

has a unique solution $u \in L_2((0,T),V)$ with weak derivative $\dot{u} \in L_2((0,T),V^*)$.

Proof. [12, Theorem 23.A] □

2.2. **Spaces and trace spaces for Maxwell's equations.** In this Section we briefly summarize the underlying space and the correct trace spaces for the eddy current problem. Therefore, in this Section let $\Omega \subset \mathbb{R}^3$ be a simply connected polyhedron with boundary Γ. The underlying Hilbert space is

$$\mathbf{H}(\mathbf{curl}, \Omega) := \{\mathbf{u} \in \mathbf{L_2}(\Omega) : \mathbf{curl}\,\mathbf{u} \in \mathbf{L_2}(\Omega)\}.$$

For the traces we fix the following notations

$$\gamma_{\mathbf{D}}\mathbf{u} := \mathbf{n}\times(\mathbf{u}|_\Gamma\times\mathbf{n}) \qquad \gamma_\times\mathbf{u} := \mathbf{u}|_\Gamma\times\mathbf{n} \qquad \gamma_{\mathbf{N}}\mathbf{u} := \mathbf{curl}\ \mathbf{u}|_\Gamma\times\mathbf{n} \qquad \gamma_{\mathbf{n}}\mathbf{u} := \mathbf{n}\cdot\mathbf{u}|_\Gamma,$$

where $\mathbf{n}$ denotes the exterior normal of Ω on the boundary Γ. For the definition of the appropriate trace spaces, please recall the definitions of the surface differential operators $\mathbf{grad}_\Gamma, \mathbf{curl}_\Gamma, \mathrm{curl}_\Gamma, \mathrm{div}_\Gamma$ (see e.g. [6, 7]). The appropriate trace spaces for polyhedral domains have been introduced by Buffa and Ciarlet in [6, 7]. The space for the Dirichlet trace $\gamma_{\mathbf{D}}$ and the Neumann trace $\gamma_{\mathbf{N}}$ are given by

$$\mathbf{H}_\perp^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) := \{\mathbf{v} \in \mathbf{H}_\perp^{-\frac{1}{2}}(\Gamma), \mathrm{curl}_\Gamma\mathbf{v} \in H^{-\frac{1}{2}}(\Gamma)\}$$
$$\mathbf{H}_\|^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{v} \in \mathbf{H}_\|^{-\frac{1}{2}}(\Gamma), \mathrm{div}_\Gamma\,\mathbf{v} \in H^{-\frac{1}{2}}(\Gamma)\}.$$

Indeed $\mathbf{H}_\perp^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$ is the dual of $\mathbf{H}_\|^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ and vice versa. The corresponding duality product is the extension of the $\mathbf{L^2_t}(\Gamma)$ duality product and in the following will be denoted with subscript τ:

$$\langle\cdot,\cdot\rangle_\tau := \langle\cdot,\cdot\rangle_{\mathbf{H}_\|^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)\times\mathbf{H}_\perp^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)}.$$

For $\mathbf{u} \in \mathbf{H}(\mathbf{curl}\,\mathbf{curl}, \mathbb{R}^3\backslash\Omega) := \{\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3\backslash\Omega) : \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} \in \mathbf{L_2}(\mathbb{R}^3\backslash\Omega)\}$ the integration by parts formula for the exterior domain $\mathbb{R}^3\backslash\Omega$ holds

$$\langle\gamma_{\mathbf{N}}\mathbf{u}, \gamma_{\mathbf{D}}\mathbf{v}\rangle_\tau = -(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{\mathbf{L_2}(\mathbb{R}^3\backslash\Omega)} + (\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}, \mathbf{v})_{\mathbf{L_2}(\mathbb{R}^3\backslash\Omega)}.$$

The Dirichlet and Neumann trace can be extended to continuous mappings.

Lemma 1 ([6, 7, 9])**.** *The trace operators*

$$\gamma_{\mathbf{D}} : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}_\perp^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$$
$$\gamma_\times : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}_\|^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$$
$$\gamma_{\mathbf{N}} : \mathbf{H}(\mathbf{curl}\,\mathbf{curl}, \Omega) \to \mathbf{H}_\|^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$$

are linear, continuous and surjective.

2.3. **Eddy current problems in conducting domains.** In this Section we collect some useful auxiliary results from the analysis of the eddy current problem in conducting regions. In this case the existence and uniqueness is well understood (see e.g. [3, 4]). Indeed the difficulty of dealing with the nonlinearity, resulting from the nonlinearity of the **B-H** curve, are discussed. Let the nonlinear operator $\mathcal{A}$ be defined by

$$\langle\mathcal{A}(\mathbf{u}), \mathbf{v}\rangle := \int_{\Omega_1} \nu_1(|\mathbf{curl}\,\mathbf{u}|)\mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v}\ \mathrm{d}\mathbf{x},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product. The main properties of $\mathcal{A}$ are summarized in the following lemma.

Lemma 2. *Let $s \mapsto \nu_1(s)s$ be strictly monotone and Lipschitz continuous and $s \mapsto \nu_1(s)$ uniformly positive and bounded, then the operator $\mathcal{A}$ is*

- *monotone, i.e.* $\langle\mathcal{A}(\mathbf{u}) - \mathcal{A}(\mathbf{v}), \mathbf{u} - \mathbf{v}\rangle \geq 0$

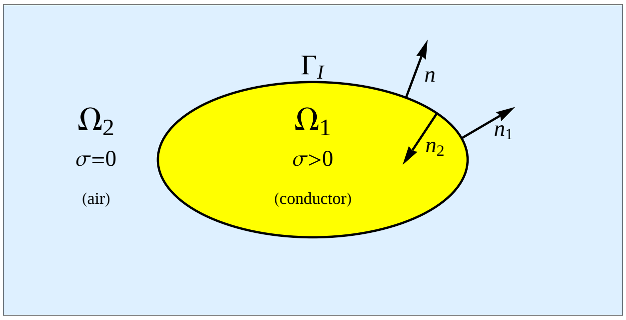

FIGURE 1. Unbounded exterior domain

- *semi-coercive, i.e.* $\langle \mathcal{A}(\mathbf{u}), \mathbf{u} \rangle \geq c \|\,\mathbf{curl}\,\mathbf{u}\|^2_{\mathbf{L_2}(\Omega)}$
- *bounded, i.e.* $\langle \mathcal{A}(\mathbf{u}), \mathbf{v} \rangle \leq c \|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl},\Omega)}$
- *hemicontinuous.*

Proof. see [3, Lemma 2.6]. □

For linear operators the whole analysis simplifies as the following remark states.

Remark 1. *Let $\mathcal{M}$ be any linear operator. If $\mathcal{M}$ is semi-coercive, i.e.* $\langle \mathcal{M}(\mathbf{w}), \mathbf{w} \rangle \geq 0$*, then $\mathcal{M}$ is also monotone.*

Hence the operator $\mathcal{A}$ resulting from the conducting part of our computational domain naturally fulfills the requirements of Theorem 1 and Theorem 2. Therefore the idea of reducing the computational domain to the conducting parts only, arises quite natural in this context.

3. The eddy current problem in $\mathbb{R}^3$

Let Ω be $\mathbb{R}^3$ and consist of two subdomains, Ω_1 and Ω_2, with the following properties. Ω_1 is Lipschitz polyhedron that is simply connected. Ω_2 is the complement of Ω_1 in $\mathbb{R}^3$, i.e $\mathbb{R}^3 \backslash \Omega_1$, and hence also simply connected. Furthermore we denote by Γ_I the interface of the two subdomains, i.e. $\Gamma_I = \overline{\Omega}_1 \cap \overline{\Omega}_2$. By $\mathbf{n}$ we denote the exterior unit normal vector field of Ω_1 on Γ_I, pointing from Ω_1 to Ω_2 (see Figure 1). Additionally to the partial differential equations in Ω_1 and Ω_2, the solution has to be sinusoidal in Ω_2. The system is completed by appropriate decay and interface conditions and an initial condition. Hence we deal with the following problem:

$$
(1) \quad \left\{
\begin{array}{rcll}
\sigma_1 \frac{\partial \mathbf{u}}{\partial t} + \mathbf{curl}\,(\nu_1(|\mathbf{curl}\,\mathbf{u}|)\,\mathbf{curl}\,\mathbf{u}) & = & \mathbf{f_1}, & \text{in } \Omega_1 \times (0,T) \\
\mathbf{curl}\,(\mathbf{curl}\,\mathbf{u}) & = & \mathbf{0} & \text{in } \Omega_2 \times (0,T) \\
\operatorname{div} \mathbf{u} & = & 0 & \text{in } \Omega_2 \times (0,T) \\
\mathbf{u} & = & \mathcal{O}(|\mathbf{x}|^{-1}) & \text{for } |\mathbf{x}| \to \infty \\
\mathbf{curl}\,\mathbf{u} & = & \mathcal{O}(|\mathbf{x}|^{-1}) & \text{for } |\mathbf{x}| \to \infty \\
\mathbf{u} & = & \mathbf{u_0} & \text{on } \Omega_1 \times \{0\} \\
\mathbf{u_1} \times \mathbf{n} & = & \mathbf{u_2} \times \mathbf{n} & \text{on } \Gamma_I \times (0,T) \\
\nu_1(|\mathbf{curl}\,\mathbf{u_1}|)\mathbf{curl}\,\mathbf{u_1} \times \mathbf{n} & = & \mathbf{curl}\,\mathbf{u_2} \times \mathbf{n} & \text{on } \Gamma_I \times (0,T)
\end{array}
\right.
$$

Here $\mathbf{u_1}$ and $\mathbf{u_2}$ are the restrictions of $\mathbf{u}$ to Ω_1 and Ω_2, i.e. $\mathbf{u_1} = \mathbf{u}|_{\Omega_1}$ and $\mathbf{u_2} = \mathbf{u}|_{\Omega_2}$.

Remark 2. *Due to scaling arguments, it can always be achieved that* $\nu_2 = 1$. *(Otherwise $\nu_1 = \nu_1/\nu_2$ and $\sigma_1 = \sigma_1/\nu_1$.)*

We show, that the degenerated parabolic problem on the whole domain (1) can be reduced to an initial value problem in the conducting region by using the tools of boundary integral operators. For the resulting parabolic equation, standard arguments provide existence and uniqueness. The crucial point in the proof is to verify the coercivity of the resulting linear or nonlinear operator and this is done in detail. The starting point of our analysis is the line-variational formulation. By Multiplying by a test function only depending on the space variable $\mathbf{x}$ and integrating over the computational domain Ω, we arrive at the following variational form.

$$\int_{\Omega_1}\left[\sigma_1\frac{\partial\mathbf{u}}{\partial t}\mathbf{v}+\nu_1(|\mathbf{curl}\,\mathbf{u}|)\mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v}\right]\mathrm{d}\mathbf{x}+\int_{\Omega_2}\nu_2\,\mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v}\,\mathrm{d}\mathbf{x}=\int_{\Omega_1}\mathbf{f_1}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}$$

Applying integration by parts in the exterior domain Ω_2 once more and using the fact, that there is no prescribed source in Ω_2, i.e. $\mathbf{curl}\;\mathbf{curl}\,\mathbf{u}=\mathbf{0}$, allows to reduce the variational problem to one just living in $\overline{\Omega}_1$.

$$\int_{\Omega_1}\left[\sigma_1\frac{\partial\mathbf{u}}{\partial t}\mathbf{v}+\nu_1(|\mathbf{curl}\,\mathbf{u}|)\mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v}\right]\mathrm{d}\mathbf{x}-\underbrace{\int_{\Gamma_I}\gamma_{\mathbf{N}}\mathbf{u}\cdot\gamma_{\mathbf{D}}\mathbf{v}\,\mathrm{dS}}_{=\langle\gamma_{\mathbf{N}}\mathbf{u},\gamma_{\mathbf{D}}\mathbf{v}\rangle_\tau}=\int_{\Omega_1}\mathbf{f_1}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}$$

In order to analyze this variational form we fix the time t. The first step is the introduction of the framework of boundary integral equations for the part corresponding to the interface Γ_I. Therefore we strongly follow the approach of Hiptmair for the frequency domain approach [9]. The boundary integral equations for the exterior problem emerge from a representation formula. In the case of Maxwell's equation this is the Stratton-Chu formula, that involves the fundamental solution of the Laplacian in three dimensions.

$$\begin{aligned}\mathbf{u}(\mathbf{x})=&\int_{\Gamma_I}(\mathbf{n}\times\mathbf{curl}\;\mathbf{u})(\mathbf{y})E(\mathbf{x},\mathbf{y})\,\mathrm{dS_y}-\mathbf{curl}_\mathbf{x}\int_{\Gamma_I}(\mathbf{n}\times\mathbf{u})(\mathbf{y})E(\mathbf{x},\mathbf{y})\,\mathrm{dS_y}\\&+\nabla_\mathbf{x}\int_{\Gamma_I}(\mathbf{n}\cdot\mathbf{u})(\mathbf{y})E(\mathbf{x},\mathbf{y})\,\mathrm{dS_y}+\int_{\Omega_2}\mathbf{curl}\;\mathbf{curl}\;\mathbf{u}(\mathbf{y})E(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}\\&-\int_{\Omega_2}\mathrm{div}\;\mathbf{u}(\mathbf{y})\nabla_\mathbf{x}E(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}.\end{aligned}\tag{2}$$

The fundamental solution of the Laplacian in three dimensions is given by

$$E(\mathbf{x},\mathbf{y}):=\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{y}|},\quad\mathbf{x},\mathbf{y}\in\mathbb{R}^3,\mathbf{x}\neq\mathbf{y}.$$

Note, that due to $\mathbf{curl}\;\mathbf{curl}\,\mathbf{u}=\mathbf{0}$ and $\mathrm{div}\,\mathbf{u}=0$, the last two terms in (2) vanish. Next we introduce the vectorial single layer potential ψ_A, the vectorial double layer potentials ψ_M and the scalar single layer potential ψ_V:

$$\begin{aligned}\psi_A(\mathbf{u})(\mathbf{x})&:=\int_{\Gamma_I}\mathbf{u}(\mathbf{y})E(\mathbf{x},\mathbf{y})\,\mathrm{dS_y}\\\psi_M(\mathbf{n}\times\mathbf{u})(\mathbf{x})&:=\mathbf{curl}_\mathbf{x}\int_{\Gamma_I}(\mathbf{n}\times\mathbf{u})(\mathbf{y})E(\mathbf{x},\mathbf{y})\,\mathrm{dS_y}\\\psi_V(\mathbf{n}\cdot\mathbf{u})(\mathbf{x})&:=\int_{\Gamma_I}(\mathbf{n}\cdot\mathbf{u})(\mathbf{y})E(\mathbf{x},\mathbf{y})\,\mathrm{dS_y}\end{aligned}$$

Taking the Dirichlet and Neumann traces of these potential operators gives rise to the definition of the boundary integral operators.

$$\begin{aligned}\mathbf{A}\lambda&:=\gamma_\mathbf{D}\psi_\mathbf{A}(\lambda)\\\mathbf{B}\lambda&:=\gamma_\mathbf{N}\psi_\mathbf{A}(\lambda)\\\mathbf{Cu}&:=\gamma_\mathbf{D}\psi_\mathbf{M}(\mathbf{u})\\\mathbf{Nu}&:=\gamma_\mathbf{N}\psi_\mathbf{M}(\mathbf{u})\\\mathbf{S}\varphi&:=\gamma_\mathbf{D}(\nabla\,\psi_V(\varphi)).\end{aligned}$$

The next theorem clarifies the continuity of the potential mappings.

Theorem 3 ([9]). *The mappings*

$$\begin{aligned}
\mathbf{A} &: \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_I) \to \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma_I)\\
\mathbf{B} &: \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_I) \to \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_I)\\
\mathbf{C} &: \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma_I) \to \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma_I)\\
\mathbf{N} &: \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma_I) \to \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_I)\\
\mathbf{S} &: H^{-\frac{1}{2}}(\Gamma_I) \to \mathbf{H}_{\perp}^{-\frac{1}{2}}(\Gamma_I)
\end{aligned}$$

are linear and bounded.

Applying the Dirichlet trace $\gamma_{\mathbf{D}}$ and the Neumann trace $\gamma_{\mathbf{N}}$ to the representation formula rewritten in terms of the potentials

$$\mathbf{u} = \psi_M[\gamma_{\mathbf{D}}\mathbf{u}] - \psi_A[\gamma_{\mathbf{N}}\mathbf{u}] - \nabla\psi_V[\gamma_{\mathbf{n}}\mathbf{u}]$$

gives rise to a Calderon mapping

$$\begin{cases} \gamma_{\mathbf{D}}\mathbf{u} = \mathbf{C}(\gamma_{\mathbf{D}}\mathbf{u}) - \mathbf{A}(\gamma_{\mathbf{N}}\mathbf{u}) - \mathbf{S}(\gamma_{\mathbf{n}}\mathbf{u}) \\ \gamma_{\mathbf{N}}\mathbf{u} = \mathbf{N}(\gamma_{\mathbf{D}}\mathbf{u}) - \mathbf{B}(\gamma_{\mathbf{N}}\mathbf{u}) \end{cases}. \tag{3}$$

Due to additional boundary term $\gamma_{\mathbf{n}}\mathbf{u}$, the extraction of the Calderon-projection is not straight forward. Heading for a Calderon-projection in a weak setting, we start by investigating the *correct* space for the Neumann trace $\gamma_{\mathbf{N}}\mathbf{u}$ (see also [9, Section 4]).

Lemma 3. *Let* $\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = 0$ *in* Ω_2 *then we have*

$$\langle \gamma_{\mathbf{N}}\mathbf{u}, \mathbf{grad}_\Gamma\varphi\rangle_\tau = 0, \quad \forall\varphi \in H^{\frac{1}{2}}(\Gamma_I).$$

Proof. By using the definition of the surface operators and Stokes-formulas on surfaces (e.g.[5, Def. 3.5 and Thm. 3.8]), we obtain

$$\begin{aligned}
\int_{\Gamma_I} \gamma_{\mathbf{N}}\mathbf{u}\cdot\mathbf{grad}_\Gamma\varphi\,\mathrm{dS} &= \int_{\Gamma_I} \mathbf{curl}\,\mathbf{u}\cdot(\mathbf{grad}_\Gamma\varphi\times\mathbf{n})\,\mathrm{dS} = \int_{\Gamma_I} \mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}_\Gamma\,\varphi\,\mathrm{dS}\\
&= \int_{\Gamma_I} \mathrm{curl}_\Gamma\,(\mathbf{curl}\,\mathbf{u})\,\varphi\,\mathrm{dS} = \int_{\Gamma_I} (\mathbf{curl}\,\mathbf{curl}\,\mathbf{u})|_{\Gamma_I}\cdot\mathbf{n}\varphi\,\mathrm{dS} = 0.
\end{aligned}$$

□

Consequently, we have, that the surface-divergence of the Neumann trace vanishes, i.e. $\mathrm{div}_\Gamma(\gamma_{\mathbf{N}}\mathbf{u}) = 0$ in a weak sense. Therefore $\gamma_{\mathbf{N}}\mathbf{u}$ is even in the gauged subspace

$$\mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma\,0, \Gamma_I) := \left\{\mu\in\mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma_I), \mathrm{div}_\Gamma\,\mu = 0\right\}.$$

The advantage of introducing this subspace is, that the following relation can be verified:

$$\langle\mu, \mathbf{grad}_\Gamma\varphi\rangle_\tau = 0, \quad \forall\mu\in\mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma\,0, \Gamma_I) \quad \forall\varphi\in H^{\frac{1}{2}}(\Gamma_I).$$

Consequently, we have more information about the impact of the additional Neumann data $\gamma_{\mathbf{n}}\mathbf{u}$

$$\langle\mu, \mathbf{S}(\varphi)\rangle_\tau = \langle\mu, \gamma_{\mathbf{D}}(\nabla\,\psi_V(\varphi))\rangle_\tau = \langle\mu, \mathbf{grad}_\Gamma\gamma_{\mathbf{D}}\psi_V(\varphi)\rangle_\tau = 0, \quad \forall\mu\in\mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma\,0, \Gamma_I).$$

Using the last identity, we can set the Calderon mapping in a weak setting. Testing with appropriate test functions μ and λ yields

$$(4)\qquad \begin{cases} \langle \mu, \gamma_{\mathbf{D}}\mathbf{u}\rangle_\tau = \langle \mu, \mathbf{C}(\gamma_{\mathbf{D}}\mathbf{u})\rangle_\tau - \langle \mu, \mathbf{A}(\gamma_{\mathbf{N}}\mathbf{u})\rangle_\tau\,, & \forall \mu \in \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma_I) \\ \langle \gamma_{\mathbf{N}}\mathbf{u}, \lambda\rangle_\tau = \langle \mathbf{N}(\gamma_{\mathbf{D}}\mathbf{u}), \lambda\rangle_\tau - \langle \mathbf{B}(\gamma_{\mathbf{N}}\mathbf{u}), \lambda\rangle_\tau\,, & \forall \lambda \in \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma_I). \end{cases}$$

In the following Lemmata we collect several properties of the boundary integral operators $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{N}$.

Lemma 4. *The bilinear form on* $\mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma_I)$ *induced by the operator* $\mathbf{A}$ *is symmetric and positive definite.*

$$\langle \lambda, \mathbf{A}\lambda\rangle_\tau \geq c_1^{\mathbf{A}} \|\lambda\|^2_{\mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)}, \qquad \forall \lambda \in \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma).$$

Proof. see [9, Thm 6.2] □

Lemma 5. *We have the symmetry property*

$$\langle \mathbf{B}(\mu), \lambda\rangle_\tau = \langle \mu, (\mathbf{C} - \mathbf{Id})(\lambda)\rangle_\tau, \qquad \forall \mu \in \mathbf{H}_{\|}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma), \lambda \in \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma).$$

Proof. see [9, Eqn. (6.5)]. □

Lemma 6. *The bilinear form on* $\mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma_I)$ *induced by the operator* $\mathbf{N}$ *is symmetric and negative semi-definite.*

$$-\langle \mathbf{N}\mu, \mu\rangle_\tau \geq c\|\mathbf{curl}_\Gamma \mu\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}, \qquad \forall \mu \in \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$$

Proof. see [9, Thm 6.4] □

The following Lemma plays an important role for providing the coercivity result, needed for proving existence and uniqueness.

Lemma 7. *We have*

$$-\langle \gamma_{\mathbf{N}}\mathbf{u}, \gamma_{\mathbf{D}}\mathbf{u}\rangle_\tau \geq 0.$$

Proof. Using the weak Calderon mapping (4) and choosing special test functions $\mu = \gamma_{\mathbf{N}}\mathbf{u}$ and $\lambda = \gamma_{\mathbf{D}}\mathbf{u}$ we obtain from the first equation and the symmetry property (Lemma 5)

$$\langle \gamma_{\mathbf{N}}\mathbf{u}, \mathbf{A}(\gamma_{\mathbf{N}}\mathbf{u})\rangle_\tau = \langle \gamma_{\mathbf{N}}\mathbf{u}, (\mathbf{C} - \mathbf{Id})(\gamma_{\mathbf{D}}\mathbf{u})\rangle_\tau = \langle \mathbf{B}(\gamma_{\mathbf{N}}\mathbf{u}), \gamma_{\mathbf{D}}\mathbf{u}\rangle_\tau\,.$$

Consequently from the second equation we obtain

$$\begin{aligned} -\langle \gamma_{\mathbf{N}}\mathbf{u}, \gamma_{\mathbf{D}}\mathbf{u}\rangle_\tau &= -\langle \mathbf{N}(\gamma_{\mathbf{D}}\mathbf{u}), \gamma_{\mathbf{D}}\mathbf{u}\rangle_\tau + \langle \mathbf{B}(\gamma_{\mathbf{N}}\mathbf{u}), \gamma_{\mathbf{D}}\mathbf{u}\rangle_\tau \\ &= -\langle \mathbf{N}(\gamma_{\mathbf{D}}\mathbf{u}), \gamma_{\mathbf{D}}\mathbf{u}\rangle_\tau + \langle \gamma_{\mathbf{N}}\mathbf{u}, \mathbf{A}(\gamma_{\mathbf{N}}\mathbf{u})\rangle_\tau\,. \end{aligned}$$

Now the result follows from the negative semi-definiteness of $\mathbf{N}$ and the positive definiteness of $\mathbf{A}$. □

Now we have provided the necessary tools for proving existence and uniqueness for the variational problem: Find $\mathbf{u} \in L_2((0,T), \mathbf{H}(\mathbf{curl}, \Omega))$ with a weak derivative $\dot{\mathbf{u}} \in L_2((0,T), \mathbf{H}(\mathbf{curl}, \Omega)^*)$, such that

$$\langle \sigma_1 \frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\rangle + \langle \mathcal{A}(\mathbf{u}), \mathbf{v}\rangle - \langle \gamma_{\mathbf{N}}\mathbf{u}, \gamma_{\mathbf{D}}\mathbf{v}\rangle_\tau = \langle \mathcal{F}, \mathbf{v}\rangle$$

for all $\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega)$. As outlined in Section 2.3, we have that $\mathcal{A}$ is positive semi-definite. Due to the positivity of the boundary term (Lemma 7), we can conclude that

$$\langle \mathcal{A}(\mathbf{u}), \mathbf{u}\rangle - \langle \gamma_{\mathbf{N}}\mathbf{u}, \gamma_{\mathbf{D}}\mathbf{u}\rangle_\tau \geq c\|\,\mathbf{curl}\,\mathbf{u}\|^2_{\mathbf{L_2}(\Omega_1)}.$$

Indeed this estimate is too weak to obtain coercivity in the full space $\mathbf{H}(\mathbf{curl}, \Omega_1)$. Even the boundary term does not fix this problem since the whole expression vanishes for $\mathbf{v} \in \mathbf{W}(\Omega_1)$. Here $\mathbf{W}(\Omega_1)$ denotes the space of gradients, given by

$$\mathbf{W}(\Omega_1) := \{\mathbf{w} = \nabla\varphi : \varphi \in H^1(\Omega_1) \text{ and } \varphi = c \text{ on } \Gamma\}.$$

In order to be able to prove coercivity in the full norm, we have to restrict $\mathbf{H}(\mathbf{curl}, \Omega_1)$ to the space of weakly divergence free functions. Consequently we introduce the gauged subspace $\bar{\mathbf{V}}$ given by

$$\bar{\mathbf{V}} := \{\mathbf{u_1} \in \mathbf{H}(\mathbf{curl}, \Omega_1) : (\mathbf{u_1}, \mathbf{w_1})_{\mathbf{L_2}(\Omega_1)} = 0, \forall\, \mathbf{w_1} \in \mathbf{W}(\Omega_1)\} \tag{5}$$

In the gauged subspace $\bar{\mathbf{V}}$ the full norm $\|\cdot\|_{\mathbf{H}(\mathbf{curl},\Omega)}$ is equivalent to the semi-norm $\|\,\mathbf{curl}\cdot\|_{\mathbf{L_2}(\Omega)}$, as the following Lemma states.

Lemma 8 (Friedrich's inequality)**.** *Let Ω_1 be a simply-connected Lipschitz domain. For all $\mathbf{u} \in \bar{\mathbf{V}}$ we have*

$$\|\mathbf{u}\|_{\mathbf{L_2}(\Omega_1)} \leq c\|\,\mathbf{curl}\,\mathbf{u}\|_{\mathbf{L_2}(\Omega_1)}.$$

Proof. see [11, Thm 3.26] □

Therefore the variational formulation restricted to this gauged subspace reads as: Find $\mathbf{u} \in L_2((0,T), \bar{\mathbf{V}})$, with a weak derivative $\dot{\mathbf{u}} \in L_2((0,T), \bar{\mathbf{V}}^*)$, such that

$$\langle \sigma_1 \frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\rangle + \langle \mathcal{A}(\mathbf{u}), \mathbf{v}\rangle - \langle \gamma_{\mathbf{N}}\mathbf{u}, \gamma_{\mathbf{D}}\mathbf{v}\rangle_\tau = \langle \mathcal{F}, \mathbf{v}\rangle \tag{6}$$

for all $\mathbf{v} \in \bar{\mathbf{V}}$. Analogous, by defining

$$\langle \mathcal{M}\mathbf{u}, \mathbf{v}\rangle := \langle \gamma_{\mathbf{N}}\mathbf{u}, \gamma_{\mathbf{D}}\mathbf{v}\rangle_\tau$$

the corresponding operator equation is given by: Find $\mathbf{u} \in L_2((0,T), \bar{\mathbf{V}})$ with a weak derivative $\dot{\mathbf{u}} \in L_2((0,T), \bar{\mathbf{V}}^*)$, such that

$$\begin{aligned} \mathcal{A}(\mathbf{u}) - \mathcal{M}\mathbf{u} &= \mathcal{F}, \quad \text{in } L_2((0,T), \bar{\mathbf{V}}^*) \\ \mathbf{u}(0) &= \mathbf{u_0}, \quad \text{in } \mathbf{L_2}(\Omega_1) \end{aligned}$$

Due to Lemma 8, we obtain coercivity in $\bar{\mathbf{V}}$. Since the operator associated to the boundary part $\mathcal{M}$ is linear, hemicontinuity and monotonicity follow due to Lemma 2. Boundedness of $\mathcal{M}$ follows by applying the trace theorems (Lemma 1. Note that $\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = \mathbf{0}$ in the exterior domain.) The precedent considerations in combination with Theorem 1 give rise to the main result of this Section:

Theorem 4. *The variational problem (6) has a unique solution $\mathbf{u} \in L_2((0,T), \bar{\mathbf{V}})$ with $\dot{\mathbf{u}} \in L_2((0,T), \bar{\mathbf{V}}^*)$.*

In the next step we show, that under additional assumptions, we even can guarantee uniqueness in the non-gauged space $L_2((0,T), \mathbf{H}(\mathbf{curl}, \Omega_1))$. We consider a test function $\mathbf{w} \in \mathbf{W}(\Omega_1)$.

$$\int_{\Omega_1} \sigma_1 \frac{\partial \mathbf{u}}{\partial t}\mathbf{w}\,\mathrm{d}\mathbf{x} + \int_{\Omega_1} \nu_1\,\mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{w}\,\mathrm{d}\mathbf{x} - \int_{\Gamma_I} \gamma_{\mathbf{N}}\mathbf{u} \cdot \gamma_{\mathbf{D}}\mathbf{w}\,\mathrm{dS} = \int_{\Omega_1} \mathbf{f_1} \cdot \mathbf{w}\,\mathrm{d}\mathbf{x}$$

Now we assume that the source $\mathbf{f_1}$ and the initial condition $\mathbf{u_0}$ are weakly divergence free, i.e.

$$\int_{\Omega_1} \mathbf{f_1} \cdot \mathbf{w}\,\mathrm{d}\mathbf{x} = 0 \quad \text{and} \quad \int_{\Omega_1} \mathbf{u_0} \cdot \mathbf{w}\,\mathrm{d}\mathbf{x} = 0, \quad \forall \mathbf{w} \in \mathbf{W}(\Omega_1). \tag{7}$$

In order to show that $\mathbf{u}(t)$ is weakly divergence free for all time t we apply Lemma 3 and recall the definition of the surface gradient $\mathbf{grad}_\Gamma(p|_\Gamma) = \gamma_{\mathbf{D}}(\nabla p), \forall p \in H^1(\Omega_1)$. So, combining these two results, for constant σ_1 we arrive at

$$\sigma_1 \int_{\Omega_1} \frac{\partial \mathbf{u}}{\partial t}\mathbf{w}\,\mathrm{d}\mathbf{x} = 0, \quad \forall \mathbf{w} \in \mathbf{W}(\Omega_1).$$

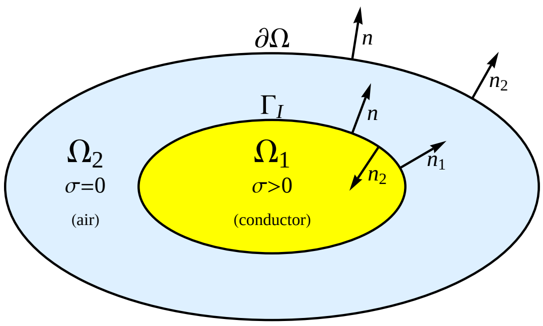

FIGURE 2. Bounded domains

Since the initial condition $\mathbf{u_0}$ is weakly divergence free, we can conclude that $\mathbf{u}(t)$ is weakly divergence-free for all t. Consequently for fixed t the solution $\mathbf{u}(t)$ is an element of the gauged space $\bar{\mathbf{V}}$, a subspace of $\mathbf{H}(\mathbf{curl}, \Omega_1)$, where we have already proven existence and uniqueness. Summarizing, if we claim the given data ($\mathbf{f_1}$ and $\mathbf{u_0}$) to be sinusoidal, then neither the $\bar{\mathbf{V}}$ gauging in the conducting domain Ω_1 nor the $\mathbf{H}_{||}^{-\frac{1}{2}}(\operatorname{div}_\Gamma 0, \Gamma)$ gauging on the interface Γ_I has to be enforced explicitly, since they are fulfilled in a natural way.

4. THE EDDY CURRENT PROBLEM IN A BOUNDED DOMAIN

In this Section the unbounded exterior domain Ω_2 is approximated by a bounded domain by introducing some artificial boundary some distance away from the conductor. Therefore let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, consisting of two subdomains Ω_1 and Ω_2, i.e. $\bar{\Omega} = \bar{\Omega}_1 \cap \bar{\Omega}_2$. Again the interface Γ_I and the normal $\mathbf{n}$ are defined in the same manner as in Section 3 (see Figure 2). Instead of appropriate decay condition, homogeneous Dirichlet boundary conditions are imposed on $\partial\Omega$. Hence we deal with the following problem:

$$
(8) \quad \left\{
\begin{array}{rcll}
\sigma_1 \frac{\partial \mathbf{u}}{\partial t} + \mathbf{curl}\,(\nu_1(|\mathbf{curl}\,\mathbf{u}|)\,\mathbf{curl}\,\mathbf{u}) & = & \mathbf{f_1}, & \text{in } \Omega_1 \times (0,T) \\
\mathbf{curl}\,(\mathbf{curl}\,\mathbf{u}) & = & \mathbf{0} & \text{in } \Omega_2 \times (0,T) \\
\operatorname{div}\mathbf{u} & = & 0 & \text{in } \Omega_2 \times (0,T) \\
\mathbf{u} \times \mathbf{n} & = & \mathbf{0} & \text{on } \partial\Omega \times (0,T) \\
\mathbf{u} & = & \mathbf{u_0} & \text{on } \Omega_1 \times \{0\} \\
\mathbf{u_1} \times \mathbf{n} & = & \mathbf{u_2} \times \mathbf{n} & \text{on } \Gamma_I \times (0,T) \\
\nu_1(|\mathbf{curl}\,\mathbf{u_1}|)\mathbf{curl}\,\mathbf{u_1} \times \mathbf{n} & = & \mathbf{curl}\,\mathbf{u_2} \times \mathbf{n} & \text{on } \Gamma_I \times (0,T)
\end{array}
\right.
$$

Again we derive the line-variational formulation. Since we impose homogeneous Dirichlet conditions on $\partial\Omega = \partial\Omega_2 \backslash \Gamma_I$, integration by parts yields the following result: For a fixed t, find $\mathbf{u} \in \mathbf{H_0}(\mathbf{curl}, \Omega)$, such that

$$
\int_{\Omega_1} \left[\sigma_1 \frac{\partial \mathbf{u}}{\partial t}\mathbf{v} + \nu_1(|\mathbf{curl}\,\mathbf{u}|)\mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v}\right] \mathrm{d}\mathbf{x} + \int_{\Omega_2} \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v}\,\mathbf{x} = \int_{\Omega_1} \mathbf{f_1} \cdot \mathbf{v}\,\mathrm{d}\mathbf{x}
$$

for all $\mathbf{v} \in \mathbf{H_0}(\mathbf{curl}, \Omega)$. Here $\mathbf{H_0}(\mathbf{curl}, \Omega)$ is the space of $\mathbf{H}(\mathbf{curl}, \Omega)$ functions with vanishing tangential trace on the boundary $\partial\Omega$, i.e.

$$
\mathbf{H_0}(\mathbf{curl}, \Omega) := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega) : \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}.
$$

Similarly to the problem in the unbounded domain, we want to reduce the full setting to a parabolic problem only settled in $\overline{\Omega}_1$. Analogous to [3] we introduce the mapping $\mathcal{H} : \mathbf{H}(\mathbf{curl}, \Omega_1) \to \mathbf{H}(\mathbf{curl}, \Omega_2)$, where $\mathbf{u_2} = \mathcal{H}(\mathbf{u_1})$ is defined as the unique solution of the problem: For given $\mathbf{u_1}$ find $\mathbf{u_2}$, such that

$$\left\{\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{u_2} &= \mathbf{0}, \quad \text{in } \Omega_2 \\ \operatorname{div}\mathbf{u_2} &= 0, \quad \text{in } \Omega_2 \\ \mathbf{u_2}\times\mathbf{n} &= \mathbf{u_1}\times\mathbf{n}, \quad \text{on } \Gamma_I \\ \mathbf{u_2}\times\mathbf{n} &= \mathbf{0}, \quad \text{on } \partial\Omega_2 \end{aligned}\right. \tag{9}$$

Here we use the notation $\mathbf{u_i} := \mathbf{u}|_{\Omega_i}$ for $i = 1, 2$.

Lemma 9. *The mapping $\mathcal{H}$ is bounded, i.e.*

$$\|\mathcal{H}(\mathbf{u_1})\|_{\mathbf{H}(\mathbf{curl},\Omega_2)} \le c\|\mathbf{u_1}\|_{\mathbf{H}(\mathbf{curl},\Omega_1)}, \quad \forall \mathbf{u_1} \in \mathbf{H}(\mathbf{curl},\Omega_1)$$

Proof. Since $\mathcal{H}(\mathbf{u_1}) = \mathbf{u_2}$ is the unique solution of (9), it is classical to deduce that the following estimate holds

$$\|\mathbf{u_2}\|_{\mathbf{H}(\mathbf{curl},\Omega_2)} \le c\|\mathbf{u_1}\times\mathbf{n}\|_{\mathbf{H}_\|^{-\frac{1}{2}}(\operatorname{div}_\Gamma,\Gamma_I)}.$$

Using the trace theorem the desired result follows:

$$\|\mathbf{u_1}\times\mathbf{n}\|_{\mathbf{H}_\|^{-\frac{1}{2}}(\operatorname{div}_\Gamma,\Gamma_I)} \le c\|\mathbf{u_1}\|_{\mathbf{H}(\mathbf{curl},\Omega_1)}.$$

□

The mapping $\mathcal{H}$ allows to define the space of **curl curl**-harmonic extended functions

$$\tilde{\mathbf{V}}_\mathbf{0} := \left\{\begin{aligned} \mathbf{u}\in\mathbf{H}(\mathbf{curl},\Omega) : &\mathbf{u_1}\in\mathbf{H}(\mathbf{curl},\Omega_1), \\ &\mathbf{u_2} = \mathcal{H}(\mathbf{u_1}), \\ &(\mathbf{u},\mathbf{w})_{L_2(\Omega)} = 0, \forall \mathbf{w}\in\mathbf{W}(\Omega_1), \\ &\mathbf{u}\times\mathbf{n} = 0 \text{ on } \partial\Omega \end{aligned}\right\}.$$

Using $\tilde{\mathbf{V}}_\mathbf{0}$, we can state the variational problem as follows: Find $\mathbf{u} \in L_2((0,T),\tilde{\mathbf{V}}_\mathbf{0})$ with weak derivative $\dot{\mathbf{u}} \in L_2((0,T),\tilde{\mathbf{V}}_\mathbf{0}^*)$, such that

$$\int_{\Omega_1}\left[\sigma_1\frac{\partial\mathbf{u}}{\partial t}\mathbf{v} + \nu_1(|\mathbf{curl}\,\mathbf{u}|)\mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v}\right]\mathrm{d}\mathbf{x} + \int_{\Omega_2}\mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v}\,\mathrm{d}\mathbf{x} = \int_{\Omega_1}\mathbf{f_1}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}$$

for all $\mathbf{v} \in \tilde{\mathbf{V}}_\mathbf{0}$. Consequently by using $\mathbf{u_2} = \mathcal{H}(\mathbf{u_1})$ we can reduce the problem to one with support only in the conducting domain $\overline{\Omega}_1$. By recalling the definition of $\bar{\mathbf{V}}$ (see (5)), we can state the variational form: Find $\mathbf{u_1} \in L_2((0,T),\bar{\mathbf{V}})$ with weak derivative $\dot{\mathbf{u}}_\mathbf{1} \in L_2((0,T),\bar{\mathbf{V}}^*)$, such that

$$\begin{aligned} &\int_{\Omega_1}\left[\sigma_1\frac{\partial\mathbf{u_1}}{\partial t}\mathbf{v_1} + \nu_1(|\mathbf{curl}\,\mathbf{u_1}|)\mathbf{curl}\,\mathbf{u_1}\cdot\mathbf{curl}\,\mathbf{v_1}\right]\mathrm{d}\mathbf{x} \\ &\qquad + \int_{\Omega_2}\mathbf{curl}\,\mathcal{H}(\mathbf{u_1})\cdot\mathbf{curl}\,\mathcal{H}(\mathbf{v_1})\,\mathrm{d}\mathbf{x} = \int_{\Omega_1}\mathbf{f_1}\cdot\mathbf{v_1}\,\mathrm{d}\mathbf{x} \end{aligned} \tag{10}$$

for all $\mathbf{v_1} \in \bar{\mathbf{V}}$.

In order to apply Theorem 1 to the variational setting (10) the crucial points are to show boundedness and coercivity of the bilinear form. Boundedness follows by the boundedness of the nonlinear part and the boundedness of the pde-harmonic extension as stated in Lemma 9. In order to show coercivity, we proceed as in the unbounded case. Since we have the non-negativity property

$$\int_{\Omega_2}\mathbf{curl}\,\mathcal{H}(\mathbf{u_1})\cdot\mathbf{curl}\,\mathcal{H}(\mathbf{u_1})\,\mathrm{d}\mathbf{x} = \|\,\mathbf{curl}\,\mathcal{H}(\mathbf{u_1})\|^2_{\mathbf{L_2}(\Omega_2)} \ge 0$$

we again obtain the estimate

$$\int_{\Omega_1}[\nu_1(|\mathbf{curl}\,\mathbf{u_1}|)\mathbf{curl}\,\mathbf{u_1}\cdot\mathbf{curl}\,\mathbf{u_1}]\,\mathrm{d}\mathbf{x} + \int_{\Omega_2}\mathbf{curl}\,\mathcal{H}(\mathbf{u_1})\cdot\mathbf{curl}\,\mathcal{H}(\mathbf{u_1})\,\mathrm{d}\mathbf{x} \ge c\|\,\mathbf{curl}\,\mathbf{u_1}\|^2_{\mathbf{L_2}(\Omega_1)}.$$

Since we imposed the restriction of weakly divergence free functions also in Ω_1, coercivity follows from Friedrich's inequality (Lemma 8). The precedent considerations give rise to the main result of this Section:

Theorem 5. *The variational problem (10) has a unique solution* $\mathbf{u_1} \in L_2((0,T),\bar{\mathbf{V}})$ *with weak derivative* $\dot{\mathbf{u}}_\mathbf{1} \in L_2((0,T),\bar{\mathbf{V}}^*)$.

Under certain additional assumptions, the solution is not only unique among the divergence-free functions, but even in the space $L_2((0,T),\mathbf{H}(\mathbf{curl},\Omega_1))$. According the approach in Section 3, we assume again that the source $\mathbf{f_1}$ and the initial condition $\mathbf{u_0}$ are weakly divergence free (cf. (7)). Now, testing (10) with $\mathbf{w} \in \mathbf{W}(\Omega_1)$ and using the fact that the **curl**-parts vanish for gradient functions, we obtain for constant σ_1

$$\sigma_1 \int_{\Omega_1} \frac{\partial \mathbf{u}}{\partial t} \mathbf{w} \, d\mathbf{x} = 0, \quad \forall \mathbf{w} \in \mathbf{W}(\Omega_1).$$

Hence $\mathbf{u}(t)$ is weakly divergence-free and consequently for fixed t an element of $\bar{\mathbf{V}}$, a subspace of $\mathbf{H}(\mathbf{curl},\Omega_1)$, where we have already proven existence and uniqueness.

5. Conclusion

We have provided existence and uniqueness results for the eddy current problem in bounded and unbounded domains. After applying appropriate boundary reduction methods, we were able to provide existence and uniqueness by standard results for evolution equations.

We want to point out two important features of our calculations. Firstly, the crucial point for proving existence and uniqueness is to ensure the non-negativity of the part related to the non-conducting domain Ω_2. The exterior domain stabilizes the coercivity since

$$-\langle \gamma_\mathbf{N} \mathbf{u_1}, \gamma_\mathbf{D} \mathbf{u_1} \rangle_\tau = \int_{\Omega_2} |\, \mathbf{curl}\ \mathcal{H}(\mathbf{u_1})|^2 d\mathbf{x} \geq 0.$$

Secondly, to avoid redundancy, the initial condition is only allowed to be prescribed in the conducting region Ω_1. Indeed, at least in the smooth case, the initial condition in the exterior domain Ω_2 is the pde-harmonic extension of the initial condition in the interior domain, i.e. the solution of the following problem

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = \mathbf{0}, & \text{in } \Omega_2 \\ \operatorname{div} \mathbf{u} = 0, & \text{in } \Omega_2 \\ \gamma_\mathbf{D} \mathbf{u} = \gamma_\mathbf{D} \mathbf{u_0}, & \text{on } \Gamma_I \end{cases}$$

with appropriate boundary or decay conditions.

The variational framework presented in this work is the starting point of various discretization techniques in time (e.g. time-stepping, multiharmonic-approaches, discontinuous Galerkin) and in space (e.g. FEM, FEM-BEM).

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