Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

Der Wissenschaftsfonds.

Fat Arcs for Algebraic Space Curves

Szilvia Béla[1] and Bert Jüttler[2]

DK3 - Geometric Solvers for Polynomial Systems,
Doctoral Program in Computational Mathematics[1],
Institute of Applied Geometry[2],

JKU Linz, Austria

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Outline

Approximating Implicitly Defined Objects

Frequently used technique in geometric computations:

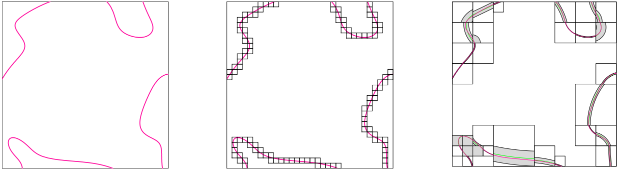
generating bounding regions

- Using the properties of the representation
  - Min-max boxes
  - Convex hulls
  - Iterative subdivision
- Using median curve/surface

  Approximating curve/surface + error bound
  - Low order approximating object (linear, quadratic)
  - Approximating object with high order (interpolation)

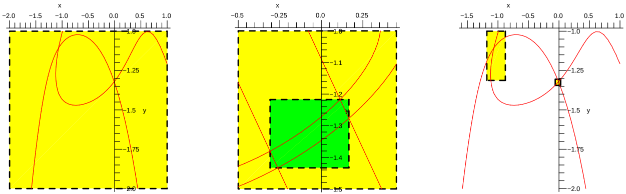
Motivation

- Implicitly defined curve approximation



Curve approximation with subdivision methods

- Solving multivariate polynomial systems



Bi-quadratic clipping

Representation of Polynomials

The multivariate functions are given by their Bernstein-Bézier tensor-product representation with respect to the domain $\Omega \subset \mathbb{R}^n$:

$$f(\mathbf{x}) = \sum_{\substack{0 \le k_i \le m_i \\ i=1..n}} d_{k_1 \dots k_n} \prod_{j=1}^{n} B_{k_j}^{m_j}(x_j).$$

Advantages:

- de Casteljau algorithm $\to$ **stable subdivision**
- variation diminishing, convex hull property $\to$ **topology detection**
- convex hull property $\to$ **fast error bound generation**

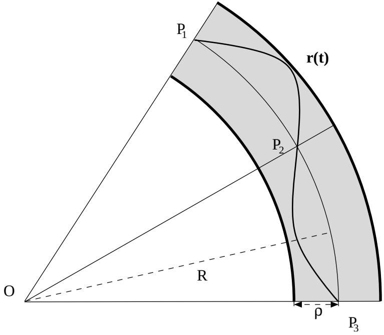
Outline

Fat Arcs for Planar Parametric Curves

Fat arc construction:

(Sederberg '89)

- Median arc generation
- Curve distance measuring
- Offset generation



Our aim:

Fat arc generation for implicitly defined curve

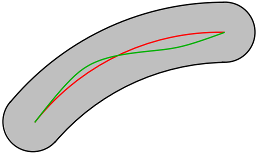
Fat Arcs

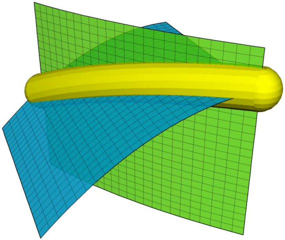
Definition

A fat arc is defined by a circular arc(median arc) $\mathcal{S}$ and a thickness $\varrho \in \mathbb{R}$.

$$\mathcal{F}(\mathcal{S},\varrho) = \{\mathbf{x} \,|\, \exists \mathbf{x}_0 \in \mathcal{S}, \quad |\mathbf{x}-\mathbf{x}_0| \leq \varrho \wedge \mathbf{x} \in \Omega \subset \mathbb{R}^n\}.$$

If $n = 2$ – the fat arc is a part of an annulus,
$n = 3$ – bounded by a segment of a torus + spherical caps.





Main Advantages of Arcs

- Cubic approximation order.
- Exact parametric and implicit representation

  $\hookrightarrow$ Parametric form: rational Bernstein-Bézier form

  - Convex hull property.

  $\hookrightarrow$ Implicit form: special quadratic equations

  - Fast computation of intersections.
  - Fast computation of the offset - boundary of the fat region.

Outline

Subdivision Method

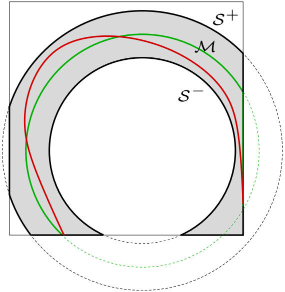
Curve: $\mathcal{C} = \{\mathbf{x} \,|\, f_i(\mathbf{x}) = 0,\ i = 1, \ldots n-1 \ \wedge\ \mathbf{x} \in \Omega \subset \mathbb{R}^n\}$
Tolerance bound: ε

- Find regions without loops or singularities

  if it fails $\rightarrow$ subdivision (until tolerance is reached)

- Fat arc generation
  - Median arc generation
  - Distance estimation

    if it fails $\rightarrow$ subdivision
    (until tolerance is reached)



- Segmentation with box boundaries.

Median Arc Generation Techniques

Curve: $\mathcal{C} = \{\mathbf{x} \,|\, f_i(\mathbf{x}) = 0,\ i = 1, \ldots n-1 \wedge \mathbf{x} \in \Omega \subset \mathbb{R}^n\}$
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  - Median arc generation
    - Interpolation
    - Least-squares approximation
    - Taylor-expansion

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Segmentation with box boundaries.

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  - Least-squares approximation
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    $\downarrow$

    Generalization

Segmentation with box boundaries.

Modification of the Taylor-expansion: 2D

Find $l(x, y) = l_0 + l_1(x - c_x) + l_2(y - c_y)$ such that

$$\text{Hess}(lf)(\mathbf{c}) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \lambda \in \mathbb{R} \tag{1}$$

in the center $\mathbf{c}$ of the domain.

Lemma

Given a polynomial f over the domain $\Omega \subset \mathbb{R}^2$. If $\|\nabla f(\mathbf{c})\| \neq 0$ in Ω, then there exists l such that $\hat{f} = lf$ satisfies (1), and it is unique up to a constant factor.

The zero set of the quadratic Taylor-expansion of $\hat{f}$ is a circle.

$$s(x, y) = T^2_{\hat{f}(c)}(x, y) = \hat{f}(\mathbf{c}) + \hat{f}_x(\mathbf{c})(x - c_x) +$$

$$+ \hat{f}_y(\mathbf{c})(y - c_y) + \lambda\left((x - c_x)^2 + (y - c_y)^2\right) \quad \forall \mathbf{x} \in \Omega.$$

Modification of the Taylor-expansion: 3D

Find two pairs of polynomials k^1, l^1 and k^2, l^2 such that

$$\begin{aligned} k^i(x,y,z) &= k_0^i + k_1^i(x-c_x) + k_2^i(y-c_y) + k_3^i(z-c_z), \\ l^i(x,y,z) &= l_0^i + l_1^i(x-c_x) + l_2^i(y-c_y) + l_3^i(z-c_z), \end{aligned}$$

and

$$\operatorname{Hess}(k^i f_1 + l^i f_2)(\mathbf{c}) = \begin{pmatrix} \lambda^i & 0 & 0 \\ 0 & \lambda^i & 0 \\ 0 & 0 & \lambda^i \end{pmatrix}, \quad \lambda^i \in \mathbb{R}, \quad i = 1,2 \qquad (2)$$

in the center $\mathbf{c}$ of the domain.

Lemma

Given f_1 and f_2 polynomials over $\Omega \subset \mathbb{R}^3$. Suppose that in the center of Ω

$$\|\nabla f_1(\mathbf{c}) \times \nabla f_2(\mathbf{c})\| \neq 0.$$

For any $(k_0^i, l_0^i) \in \mathbb{R}^2 \backslash \{(0,0)\}$ there exists a non-trivial solution for k^i and l^i such that $k^i f_1 + l^i f_2$ satisfies (2), and it is unique up to a constant factor.

Arc Generation - 3D Taylor-expansion

We choose (k_0^1, l_0^1) and (k_0^2, l_0^2)

$$\hat{f}_1 = k^1 f_1 + l^1 f_2 \quad \text{and} \quad \hat{f}_2 = k^2 f_1 + l^2 f_2.$$

The quadratic Taylor expansions have spherical zero level set:

$$s_1 = T^2_{\hat{f}_1(\mathbf{c})}(\mathbf{x}) \quad \text{and} \quad s_2 = T^2_{\hat{f}_2(\mathbf{c})}(\mathbf{x})$$

The median arc $\mathcal{S}$ is the zero set of the polynomials s_1 and s_2

$$\mathcal{S} = \{(\mathbf{x}) : s_1(\mathbf{x}) = 0 \wedge s_2(\mathbf{x}) = 0 \wedge \mathbf{x} \in \Omega\}.$$

Convergence of the Approximating Arc

Zero set of circular/spherical equation(s)

$\downarrow$

This can be used as the median arc.

Why is it a good choice?

If $\mathbf{c}$ is a point from the approximated plane curve, then the computed median arc is the osculating circle of the algebraic curve in $\mathbf{c}$.

If $\mathbf{c}$ is a point from the approximated space curve, then the computed sphere family has one common intersection arc which is the osculating circle of the algebraic curve in $\mathbf{c}$.

$\downarrow$

Convergence.

Distance Estimation

Curve: $\mathcal{C} = \{\mathbf{x} \,|\, f_i(\mathbf{x}) = 0,\, i = 1, \ldots n-1 \,\wedge\, \mathbf{x} \in \Omega \subset \mathbb{R}^n\}$
Tolerance bound: ε

- Find regions without loops or singularities

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- Fat arc generation

  Median arc generation

  - Distance estimation

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    - One-sided Hausdorff-distance
    - Convex hull property

  Segmentation with box boundaries.

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  Segmentation with box boundaries.

- One-sided Hausdorff-distance
- Convex hull property

$\downarrow$

Generalization

Distance Bound for Algebraic Curves

Consider the distance bound of f and s over the domain $\Omega \subset \mathbb{R}^n$:

$$\mathbf{d}(f, s) = \|f(\mathbf{x}) - s(\mathbf{x})\|_{BB}^{\Omega}.$$

2D

$$\begin{array}{lcll} \text{The curve:} & \mathcal{C} & = & \{\mathbf{x} \,|\, f(\mathbf{x}) = 0 \wedge \mathbf{x} \in \Omega\}. \\ \text{The median arc:} & \mathcal{S} & = & \{\mathbf{x} \,|\, s(\mathbf{x}) = 0 \wedge \mathbf{x} \in \Omega\}. \end{array}$$

If there is $0 < G \leq \|\nabla f\|$, then the algebraic curve distance bounded by

$$\varrho = \frac{\mathbf{d}(f, s)}{G}.$$

The fat domain defined by:

$$\mathcal{F}(\mathcal{S}, \varrho) = \{\mathbf{x} \,|\, \exists \mathbf{x}_0 \in \mathcal{S},\, |\mathbf{x}_0 - \mathbf{x}| \leq \varrho\}.$$

$$\boxed{\mathcal{C} \subseteq \mathcal{F}}$$

Distance Bound for Algebraic Curves

3D - with using orthogonalized combination of $\hat{f}_1(\mathbf{x})$ and $\hat{f}_2(\mathbf{x})$

$$\begin{array}{lrcl} \text{The curve:} & \mathcal{C} & = & \{\mathbf{x} \,|\, \tilde{f}_1(\mathbf{x}) = 0 \wedge \tilde{f}_2(\mathbf{x}) = 0 \ \wedge \mathbf{x} \in \Omega\}. \\ \text{The median arc:} & \mathcal{S} & = & \{\mathbf{x} \,|\, \tilde{s}_1(\mathbf{x}) = 0 \wedge \tilde{s}_2(\mathbf{x}) = 0 \ \wedge \mathbf{x} \in \Omega\}. \end{array}$$

If there exist $0 < G$, $0 < K$ such that

$$G \leq \|\nabla f_1\|, \quad G \leq \|\nabla f_2\|, \quad |\nabla f_1 \cdot \nabla f_2| \leq K \quad \text{and} \quad 0 < G^2 - K,$$

then the algebraic curve distance bounded by

$$\varrho = \sqrt{\frac{\mathbf{d}(\tilde{f}_1, \tilde{s}_1)^2 + \mathbf{d}(\tilde{f}_2, \tilde{s}_2)^2}{G^2 - K}}.$$

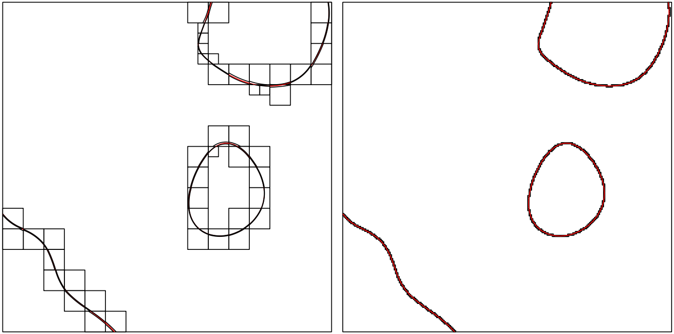
The fat arc is the point set

$$\mathcal{F}(\mathcal{S}, \varrho) = \{\mathbf{x} \,|\, \exists \mathbf{x}_0 \in \mathcal{S}, \ |\mathbf{x}_0 - \mathbf{x}| \leq \varrho\}.$$

$$\boxed{\mathcal{C} \subseteq \mathcal{F}}$$

Example - 2D

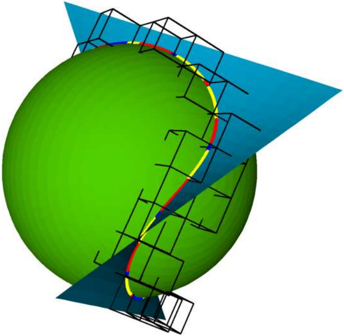
Fat arcs Boxes



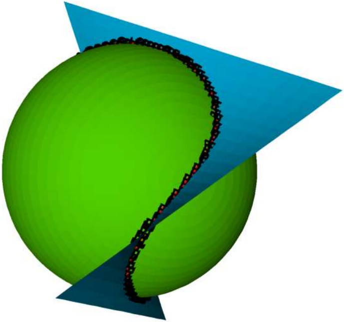
Approximation with arcs (Taylor-expansion and bounding boxes).
In the unit box with tolerance bound: 0.01.
Number of bounding primitives: 46 and 685.

Example - 3D

Fat arcs



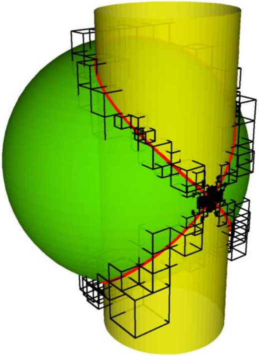
Boxes



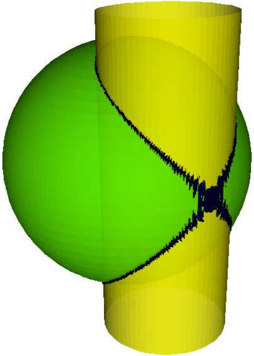
Approximating arcs for an implicitly defined space curve.
In the unit box with tolerance bound: 0.05.
Number of bounding primitives: 35 and 319.

Example - 3D

Fat arcs



Boxes



Generate bounding domains around a singularity.
In the unit box with tolerance bound: 0.05.
Bounding primitives: 277 fat arcs + 22 boxes and 1320 boxes.

Outline

Iterative Domain Reduction

Polynomial system: $\mathcal{R} = \{\mathbf{x} \,|\, f_i(\mathbf{x}) = 0,\, i = 1, \ldots n \wedge \mathbf{x} \in \Omega \subset \mathbb{R}^n\}$
Tolerance bound: ε

- Find segments without singular points

  if it fails $\rightarrow$ subdivision (until tolerance is reached)

- Fat arc/sphere generation for each implicit curve/surface
  - Median arc/sphere generation
  - Distance estimate

  if it fails $\rightarrow$ subdivision (until tolerance is reached)

- Generate min-max box around the intersection

Fat Arc Generation 2D

Find

$$\mathcal{R} = \{(x,y) \,|\, f_i(x,y) = 0,\ i = 1,2 \wedge (x,y) \in \Omega \subset \mathbb{R}^2\}.$$

We compute $\hat{f}_i$ with the modification of the Taylor-expansion.

$$\hat{f}_1 = l_1 f_1 \text{ and } \hat{f}_2 = l_2 f_2.$$

$$s_1 = T^2_{\hat{f}_1}(x_c, y_c) \text{ and } s_2 = T^2_{\hat{f}_2}(x_c, y_c).$$

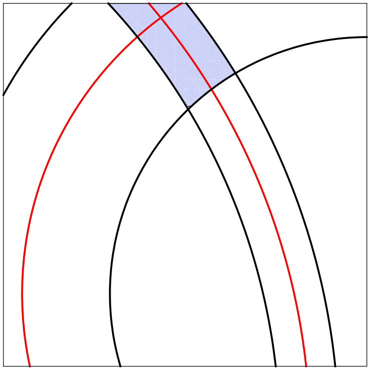
Using the former distance estimation for BB-polynomials:

$$\mathcal{B} = \mathcal{F}(\hat{f}_1, s_1) \cap \mathcal{F}(\hat{f}_2, s_2) \cap \Omega$$

$$\downarrow$$

$$\boxed{\mathcal{R} \subseteq \mathcal{B}}$$

Fat Arc Generation - Example



The intersection of the fat arcs is bounded by a curved polygon.

Fat Sphere Generation 3D

Find

$$\mathcal{R} = \{(x, y, z) \,|\, f_i(x, y, z) = 0,\ i = 1..3 \wedge (x, y) \in \Omega \subset \mathbb{R}^3\}.$$

We compute $\hat{f}_i$ with the modification of the Taylor-expansion.

$$\hat{f}_1 = k_1 f_1 + l_1 f_2,\ \hat{f}_2 = k_2 f_1 + l_2 f_3 \text{ and } \hat{f}_3 = k_3 f_2 + l_3 f_3.$$

Choose an element of each sphere family:

$$s_i = T^2_{\hat{f}_i}(x_c, y_c, z_c), \quad i = 1...3.$$

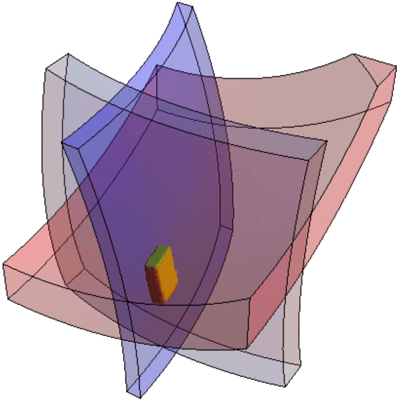
Using the former distance estimation:

$$\mathcal{B} = \mathcal{F}(\hat{f}_1, s_1) \cap \mathcal{F}(\hat{f}_2, s_2) \cap \mathcal{F}(\hat{f}_3, s_3) \cap \Omega$$

$$\downarrow$$

$$\boxed{\mathcal{R} \subseteq \mathcal{B}}$$

Fat Sphere Generation - Example



The intersection of the fat spheres is bounded by plane segments and spherical patches.

Domain Reduction

Topology detection

to find single segments from both implicitly defined functions

if it fails $\rightarrow$ subdivision

Fat arc/sphere generation

Median arc/sphere generation

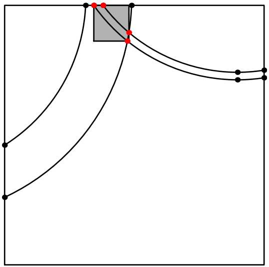
Distance estimate

Intersection

if it fails $\rightarrow$ subdivision

- Generate min-max box around the intersection

Min-max Box Generation



The min-max box around the intersection is the min-max box around

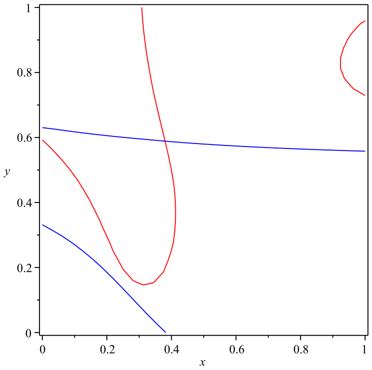
- intersection points of fat region boundaries
- intersection points of fat region boundaries and box boundaries contained by the other fat regions
- points on fat region boundaries with normal vector pointing into the direction of a coordinate axis.

Example 1. 2D

the degree of f : $(4, 7)$

the degree of g : $(3, 2)$

$$\boxed{\begin{array}{l} 7.07107\,10^{-1} \\ 1.59900\,10^{-2} \\ 4.66016\,10^{-7} \\ 1.61630\,10^{-20} \\ 1.90841\,10^{-60} \end{array}}$$

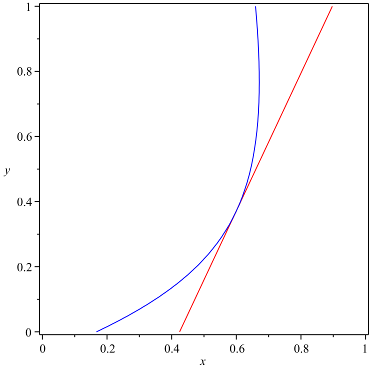


Example 2. - 2D Double root

the degree of f : $(3,3)$

the degree of g : $(2,2)$

$$\boxed{\begin{array}{c} 1/2\sqrt{2} \\ 6.62919\,10^{-2} \\ 9.11414\,10^{-3} \\ 3.85377\,10^{-4} \\ 3.57993\,10^{-6} \\ 3.35420\,10^{-9} \end{array}}$$



Conclusion and future work

Algorithms:

- bound algebraic curves in 2 and 3 dimensional space
- bound the root of polynomial systems with two or three variables
- theoretical analysis of the rate of convergence

Future work:

- generalization for higher dimensional problems
- interval-type methods for robust computation

Thank you for your attention!