Maker–Breaker domination number

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2022.04.05. SLC 87

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FWF Der Wissenschaftsfonds.

[1]The author is supported by the Austrian Science Fund (FWF): W1214-N15, project DK9.

MBD game — what game?

- Given a graph $G = (V, E)$, two players take turns to claim a vertex from V that is unclaimed yet.
- One is called Dominator (the maker), the other is called Staller (the breaker).
- Dominator wins if all the vertices he has claimed form a domination set of G.
- Staller wins if she prevents Dominator from winning, that is to claim some vertex and all its neighbors, so that Dominator cannot dominate that vertex with any of his claimed vertices.
- This game is called the *Maker–Breaker domination game*, abbreviated as *MBD game*.

Maker–Breaker domination number

- The minimum number of moves for Dominator to guarantee his winning of the game on a given graph is an invariant for the graph.
- This number is denoted by $\gamma_{MB}(G)$ when Dominator is the first one to play, and by $\gamma'_{MB}(G)$ when Dominator is the second to play.
- Note that this number is usually denoted as ∞ if Dominator does not have any winning strategies for the game; this number is finite when he has a winning strategy.
- *Domination number* of a graph G is the cardinality of its minimum-sized domination set(s), denoted by $\gamma(G)$; such sets are called γ*-sets*.

Today's content?

- First, we investigate a base case for the graph structure when its Maker–Breaker domination number equals to its domination number.
- Second, we figure out the domination number for $P_2 \square P_n$.

Which graphs have this property?

It is obvious that $\gamma_{MB}(G) = 1$ if and only if $\gamma(G) = 1$.

Theorem

Let G be a graph with $\gamma(G) = 2$. Then $\gamma_{MB}(G) = \gamma(G)$ if and only if G is a spring graph with 2 groups.

spring graph?

- Let $Q_1 := \{a\}$, $Q_i := \{b_i, c_i\}$ for $2 \leq i \leq k$. Note that $a, b_2, c_2, \ldots, b_k, c_k$ are pairwise distinct vertices. Denote by $Q := \cup_{i=1}^{k} Q_i$.
- Let A_i be a set of vertices such that $A_i \cap Q_j = \emptyset$ for all $1 \leq i, j \leq k$. Denote by $A := \cup_{i=1}^{k} A_i$.
- We see that $A \cap Q = \emptyset$.
- Let G be a graph such that $V(G) = Q \dot{\cup} A$, and $E(G)$ be such that any vertex in A_i is adjacent to all vertices in Q_i, and that either $\{b_i, c_i\} \in E(G)$ or b_i is adjacent to all vertices in Q_j, c_i is adjacent to all vertices in Q_k for some $j, k \neq i$.

spring graph

- If a graph can be obtained from the described process, we call it a *spring graph with* k *groups*.
- We say that graph $G_2 = (V_2, E_2)$ is an *expansion* of graph $G_1 = (V_1, E_1)$ if $V_1 = V_2$ and $E_1 \subset E_2$.
- We see that any expansion of a spring graph with k groups is still a spring graph with k groups.

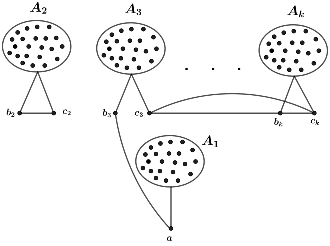

Figure: This is an illustration of a minimal spring graph.

spring graph and its domination sets

Theorem

Let $\mathcal{G}$ be a graph with $\gamma(\mathcal{G}) = k \geq 2$. Then the following two statements are equivalent:

1. *$\mathcal{G}$ has at least 2^{k-1} γ-sets and each of them has the form $\{a, \overline{Q_2}, \cdots, \overline{Q_k}\}$, where $\overline{Q_i}$ $(2 \leq i \leq k)$ represents one element in set $Q_i = \{b_i, c_i\}$.*
2. *$\mathcal{G}$ is a spring graph with k groups.*

proof: $2 \Rightarrow 1$

- Let $\mathcal{G}$ be a spring graph with k groups, and with $\gamma(\mathcal{G}) = k \geq 2$.
- It is not hard to see that there are 2^{k-1} many such sets, since each $\overline{Q_i}$ $(2 \leq i \leq k)$ has two choices.
- We only need to show that such set is indeed a γ-set of $\mathcal{G}$.
- Let $S := \{a, b_2, \cdots, b_k\}$.
- From the structure of $\mathcal{G}$, we know that c_i is either adjacent to b_i, or to all the vertices in some Q_j $(j \neq i)$, which says that c_i is adjacent to b_j or a (when $j = 1$).
- Therefore, any vertex of $\mathcal{G}$ that are not in S has a neighbor in S. This implies that S is a dominating set of $\mathcal{G}$.
- Since $|S| = k = \gamma(\mathcal{G})$, we know that S is a γ-set of $\mathcal{G}$.
- The other cases when $\overline{Q_i} = c_i$ for some $2 \leq i \leq k$ can be argued analogously.

proof: $1 \Rightarrow 2$

- Let $\mathcal{G}$ be a graph fulfilling the condition described in item 1. Recall that $Q = \{a, b_2, c_2, \ldots, b_k, c_k\}$.
- Consider the dominating set $D = \{a, b_2, \ldots, b_k\}$. Assume c_i is not adjacent to a or b_i. Then we substitute $b_j \in D$ such that b_j is adjacent to c_i to c_j, forming another dominating set D'.
- Then, there exists $l \neq i$ such that c_i is adjacent to both c_l and b_l, since D' is a dominating set. Therefore, c_i must be adjacent to either a or b_i or both vertices in Q_l $(l \neq i)$.
- Because of the general symmetry of b_i and c_i, we get the following requirements on the edges between vertices in Q:
  - either $\{b_i, c_i\} \in E(G)$ or b_i is adjacent to all vertices in Q_j, c_i is adjacent to all vertices in Q_k for some $j, k \neq i$.

proof: $1 \Rightarrow 2$

- Let $v \in V(\mathcal{G}) \setminus Q$. Suppose there exists $q_i \in Q_i$ such that v is not adjacent to q_i for each $1 \leq i \leq k$.
- Consider the dominating set $\{q_1 = a, q_2, \ldots, q_k\}$.
- Vertex v is not dominated by any vertex in this set, which leads to a contradiction.
- Hence, there exists Q_i such that v is adjacent to all vertices in Q_i. Then we put vertex $v \in V(\mathcal{G}) \setminus Q$ into group A_i.
- So far we have proved that $\mathcal{G}$ is an expansion of some minimal spring graph with k groups, hence is also a spring graph with k groups.

open problem

- we gave the structural characterization for the graphs G with $\gamma(G) = \gamma_{MB}(G) = k = 2$.
- We introduced spring graphs for the description and gave an equivalent description in the view of domination sets for such graphs.
- It would be interesting to see if these results can be of any help when $k > 2$.

$P_2 \square P_n$

- Let $V_n = \{u_1, ..., u_n, v_1, ..., v_n\}$.
- Let $E_n = \{\{u_i, u_{i+1}\} \mid i = 1, 2, ..., n-1\} \cup \{\{v_i, v_{i+1}\} : i = 1, 2, ..., n-1\} \cup \{\{u_i, v_i\} : i = 1, 2, ..., n\}$.
- $P_2 \square P_n := (V_n, E_n)$.

$\gamma'_{MB}(P_2 \square P_n)$

Theorem

$\gamma'_{MB}(P_2 \square P_n) = n$ for $n \geq 1$, and Dominator cannot skip any moves, otherwise he cannot win.

- For $\gamma'_{MB}(P_2 \square P_n) \leq n$, we propose the "pairing strategy".
- The pairing vertices are $\{v_i, u_i\}$, there are n pairs.
- Whenver Staller chooses one vertex, let Dominator choose its pairing vertex.
- In this way, he can win within n rounds.
- For the other direction, we need a proposition first.

$\gamma'_{MB}(P_2 \square P_n)$

Proposition

$\gamma_{MB}(\rho_m) = m$ for $m \geq 0$; when $m \geq 2$, Dominator will not skip any move, otherwise he would lose the game.

By ρ_m ($m \geq 0$) denote the status of the graph $P_2 \square P_m$ during the MBD game. This is when v_2 is already claimed by Staller and u_1 is already dominated by Dominator.

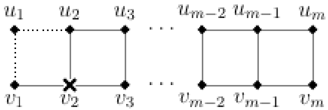

Figure: This is an illustration of the graph ρ_m.

$\gamma'_{MB}(P_2 \Box P_n)$

Theorem

$\gamma'_{MB}(P_2 \Box P_n) = n$ for $n \geq 1$, and Dominator cannot skip any moves, otherwise he cannot win.

Proof.

For the other direction, it is easy to argue when $n = 1$. When $n = 2$, let Staller choose u_2 for the first step. Then we see that the remaining game is harder for Dominator to win, in comparison with playing the game on the graph ρ_n. By the above proposition, we know that Dominator needs n steps to win on ρ_n, and he cannot skip any moves. Hence $\gamma'_{MB}(P_2 \Box P_n) \geq n$, and Dominator cannot skip any moves. □

$\gamma_{MB}(P_2 \square P_n)$

Theorem

$\gamma_{MB}(P_2 \square P_n) = n - 2, \ n \geq 13.$

For one direction, we need the following theorem.

Theorem

$\gamma_{MB}(P_2 \square P_{13}) = 11.$

$\gamma_{MB}(P_2 \square P_n)$

Theorem

$\gamma_{MB}(P_2 \square P_n) = n - 2,\ n \geq 13.$

- Consider the graph as the two subgraphs $A := P_2 \square P_{13}$ and $B \cong P_2 \square P_{n-13}$ connected by edges $\{u_{13}, u_{14}\}$, $\{v_{13}, v_{14}\}$, up to isomorphism.
- Let Dominator start the move on A and respond on A whenever Staller claims a vertex of A, using the strategy for $P_2 \square P_{13}$; and let Dominator respond on B with the pairing strategy whenever Staller claims a vertex of B.
- In this way, we see that he needs within $11 + (n - 13) = n - 2$ steps in total, in order to win. Therefore, $\gamma_{MB}(P_2 \square P_{13}) \leq n - 2,\ n \geq 13.$
- For the other direction, we need the following theorem first.

$\gamma_{MB}(P_2\Box P_n)$

Theorem

$\gamma_{MB}(X_m) \geq m - 2$ *for* $m \geq 1$.

By X_m ($m \geq 1$) denote the status of graph $P_2\Box P_m$ during the MBD game. This is when u_1 is already dominated by Dominator.

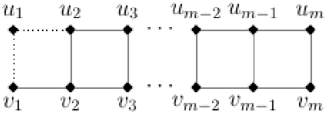

Figure: This is an illustration of the graph X_m.

$\gamma_{MB}(P_2 \square P_n)$

Theorem

$\gamma_{MB}(P_2 \square P_n) = n - 2, \ n \geq 13.$

Proof.

For the lower bound, $\gamma_{MB}(P_2 \square P_n) \geq \gamma_{MB}(X_n)$ since $P_2 \square P_n$ has one more un-dominated vertex than the graph X_n. Hence $\gamma_{MB}(P_2 \square P_n) \geq \gamma_{MB}(X_n) \geq n - 2$ when $n \geq 13$. To conclude, $\gamma_{MB}(P_2 \square P_n) = n - 2, \ n \geq 13.$ □

open problem

- The exact result for general cartesians $P_m \Box P_n$ does not seem easy.
- It would be interesting to consider the situation for $P_3 \Box P_n$, as a starting point.

reference

J. Forcan and J. Qi.
Maker–Breaker domination number for $P_2 \Box P_n$.
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Thank You