How to avoid collision of 3D-realization for moving graphs

Jiayue Qi

Doctoral Program Computational Mathematics (DK)

RISC, Johannes Kepler University, Linz

Conference on Geometry: Theory and Applications, CGTA, 2019-06-06

motivation

- See illustration of two Lego models.
- If we look at the Lego models in my hand from above. We see a “moving graph” in $\mathbb{R}^2$.
- The difference between these two models are: One can accomplish 360-degree full motion; the other has collision problem and cannot accomplish the full motion.

motivation

- Problem to be focused on: **Given a moving graph M, how to find a collision-free L-model for it.**
- In the problem statement there are several items not yet defined: moving graph, L-model, collision-free L-model.
- Don't worry, definitions come now!

moving graph

Definition (moving graph)

A moving graph is a pair $M = (G, F)$, where $G = (V, E)$ is a graph and $F = \{f_v : \mathbb{R} \to \mathbb{R}^2, t \mapsto (x_v(t), y_v(t)) | v \in V, \|f_{v_i}(t) - f_{v_j}(t)\|$ is a constant if $v_i v_j \in E\}$.

- We assign a "paramitrized position" to every vertex such that every edge length stays a constant.

collision pair

- If we look again to this Lego model, trying to find the situation when these edges are not yet set into different layers. (Look from above, consider it as a moving graph.)
- We can see that for a moving graph, collisions happen and only happen when a vertex collides with an edge.

Definition (collision in a moving graph)

Let $M = (G, F)$ be a moving graph, where $G = (V, E)$. Vertex $v_k \in V$ collides with edge $v_i v_j \in E$ if and only if the equation $\|f_{v_k}(t) - f_{v_i}(t)\| + \|f_{v_k}(t) - f_{v_j}(t)\| = \|f_{v_i}(t) - f_{v_j}(t)\|$ has solution(s) in $\mathbb{R}$, where $f_{v_k}, f_{v_i}, f_{v_j} \in F$ and then $(v_k, v_i v_j)$ is called a **collision pair** in M.

- The equation in the above definition says nothing more than $|v_i - v_k| + |v_j - v_k| = |v_j - v_i|$ where $v_i v_j \in E$ has solution.
- "vertex hitting edge"

L-model

Inspired by “Lego models”, we define “L-models”.

Definition (L-model)

Let $M = (G, F)$ be a moving graph, where $G = (V, E)$. An L-model of M is a pair $L = (M, h)$, where $h : E \to \mathbb{Z}$ is a function assigning to each edge of G an integer height value.

- Basicly, here we assign to every edge of the moving graph a height value.

collision-free L-model

- If we observe the model again, we can see that some collisions in the moving graph can be avoided with the help of this height function setting.
- However, some collisions may still happen in the 3D-realization for our L-model.
- And a collision happen in L-model if and only if the height of edge e is within the height range of vertex v for some collision pair (v, e).
- This is because of this specific type of joint we have here in our model. Every vertex is represented by one "vertically-consecutive" Lego stick.

collision-free L-model

Definition (collision-free L-model)

Let $L = (M, h)$ be an L-model, where $M = (G, F)$ and $G = (V, E)$. L is collision-free if and only if

$$h(v_i v_j) \notin [\min_{vv_k \in E} h(vv_k), \max_{vv_k \in E} h(vv_k)]$$

for any collision pair $(v_k, v_i v_j)$ in M.

Figure: Photo of a 3D-realization of one L-model.

Problem statement

Given a moving graph M, how to find a collision-free L-model for it.

collision-detection

- Idea: For all vertices, check for all edges according to the definition if they can form a collision pair with the vertex.
- implemented in Mathamatica
- So after we get the information of collision pairs, how should we proceed?
- We want to assign to each edge a height value, hoping those collisions can be avoided when edges are set in some different layers.
- In order to state our method better, we need more definitions first..

more definitions

Definition (collision graph)

Let $M = (G, F)$ be a moving graph, where $G = (V, E)$. The collision graph of M, denoted as graph C, is $C = (V_C, E_C)$, where $V_C = E$ and $\overrightarrow{e_i e_j} \in E_C$ if and only if at least one of the vertices of e_i collide(s) with edge e_j in M.

- Note that vertex set of collision graph is the edge set of the moving graph.
- Note that the collision graph is a directed graph.

Definition (induced collision graph)

For a moving graph $M = (G, F)$, where $G = (V, E)$. $S \subset E$, the collision graph of M induced by S is $C[S]$, the subgraph of C induced by S, where C is the collision graph of M.

example

Example

Consider one example from the class of Dixon-1 moving graphs, denote it as M, $M = (G, F)$, $G = (V, E)$. It is a complete bipartite graph with 7 vertices. Two independent sets of V are $\{1, 2, 3, 4\}$ and $\{5, 6, 7\}$.

Functions in F are:

$f_1(t) = (\sin t, 0)$,

$f_2(t) = (\sqrt{1 + \sin^2 t}, 0)$,

$f_3(t) = (-\sqrt{2 + \sin^2 t}, 0)$,

$f_4(t) = (\sqrt{3 + \sin^2 t}, 0)$,

$f_5(t) = (0, \cos t)$,

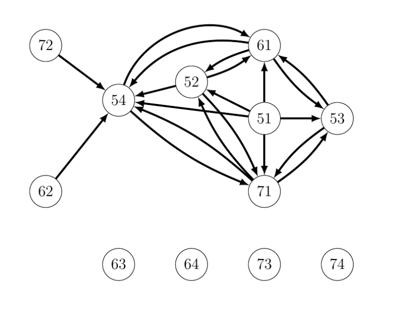
$f_6(t) = (0, \sqrt{1 + \cos^2 t})$,

$f_7(t) = (0, -\sqrt{2 + \cos^2 t})$.

example

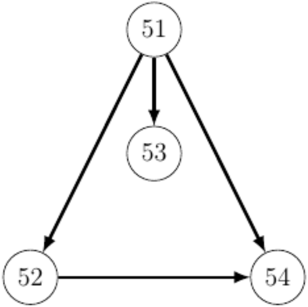
Example

Apply the collision-detection program, we get the collision pairs in M:
$(1, 52), (1, 53), (1, 54), (2, 54), (5, 61), (5, 71)$.
From collision information, we construct the collision graph C.



example

- So what is the "collision graph induced by $S = \{51, 52, 53, 54\}$"?
- By definition, it is the subgraph of graph C induced by S, i.e.,



Sufficient condition for the existence of a collision-free L-model

Theorem

Let $M = (G, F)$ be a finite moving graph, where $G = (V, E)$. If there exists a partition of E into two parts E_L and E_U, such that the induced collision graphs of M, $C_L = C[E_L]$ and $C_U = C[E_U]$ both are acyclic, then there exists a collision-free L-model of M.

We also give an algorithm constructing the height function for L-model, when this condition is fulfilled. I will show it with the coming example.

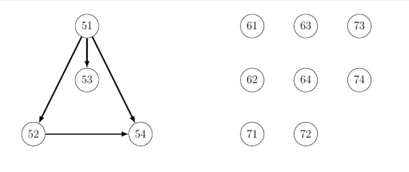
example

Example

We partition edges of M into two parts:

$$E_U = \{51, 52, 53, 54\}, E_L = \{61, 62, 63, 64, 71, 72, 73, 74\},$$

then from this two parts we construct the induced collision graphs C_U and C_L, respectively.



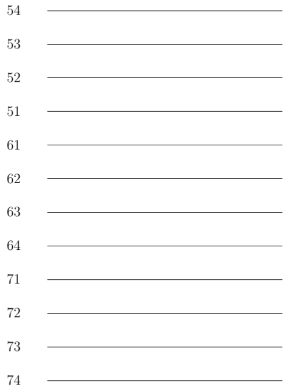
Both C_L and C_U are acyclic, so there exists a collision-free L-model for M.

algorithm

- Since both induced collision graphs are acyclic, there is a natrual partial order.
- We start by randomly choosing one part to be the "upper part", say $\{51, 52, 53, 54\}$-part.
- Then under the partial order, there is a smallest element, we set the height of it to be 1, i.e., $h(51) = 1$.
- Then we delete this element and its corresponding edges. Find the smallest element, set its height to be 2, i.e., $h(52) = 2$.
- Repeat this process.
- Note that for the other part ("lower part") the height value goes lower, from 0 to -1, etc.

algorithm

- After applying the algorithm stated above, we get height function for our graph M as below.
- Recall that collision pairs are $(1,52),(1,53),(1,54),(2,54),(5,61),(5,71)$. Check!



converse part does not hold

Theorem

There exists a moving graph $M = (G, F)$ where $G = (V, E)$ that has a collision-free L-model but we cannot partition E into two parts such that the induced collision graph of M in both parts are acyclic.

Remark

Counterexample: S_2 moving graph.
https://jan.legersky.cz/project/movable_graphs_animations/special/

Dixon1 moving graph

- They are isomorphic to $K_{m,n}$, the complete undirected bipartite graph with $m+n$ vertices.
- Fix real numbers $0 < a_1 < \ldots < a_{m-1}$ and $0 < b_1 < \ldots < b_{n-1}$, $a_0 = b_0 = 0$.
- Vertices: $p_0, \ldots, p_{m-1}$ and $q_0, \ldots, q_{n-1}$.
- parametrization:
  $p_0 = (\sin t, 0)$,
  $p_i = (\pm\sqrt{a_i + \sin^2 t}, 0)$, for $i = 1, 2, \ldots, m-1$
  $q_0 = (0, \cos t)$, for $j = 1, 2, \ldots, n-1$
  $q_j = (0, \pm\sqrt{b_j + \cos^2 t})$.

Dixon1 moving graph

See animation of a Dixon1 moving graph. https://jan.legersky.cz/project/movable_graphs_animations/dixon/

Question: Is it possible to realize a collision-free L-model for it?

Theorem

Every Dixon-1 moving graph has a collision-free L-model.

See 3D animation of a Dixon-1 moving graph. `https://jan.legersky.cz/project/movable_graphs_animations/dixon/`

Dixon-2 moving graph

The vertex-set is : $\{1,2,3,4,5,6,7,8\}$. It's $K_{4,4}$ and two independent sets of vertex-set are $\{1,2,3,4\}, \{5,6,7,8\}$.
See animation: `https://jan.legersky.cz/project/movable_graphs_animations/dixon/`

Dixon-2 moving graph

Theorem

There is no collision-free L-model for Dixon-2 moving graphs.

Thank You