A calculus for monomials in Chow group $A^{n-3}(n)$

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ambient ring, problem statement

- Ambient ring: Chow ring of M_n ($n \in \mathbb{N}$, $n \geq 3$). Denote it as $A^*(n)$.
- M_n: moduli space of stable n-pointed curves of genus zero.
- A bipartition $\{I, J\}$ of N where the cardinality of both I and J are at least 2 is called a **cut** (of M_n).
- Denote $\delta_{I,J}$ as the class of the cut subvariety $D_{I,J}$ of M_n.
- It's a graded ring: $A^*(n) = \bigoplus_{k=0}^{n-3} A^k(n)$; $A^r(n)$ is a **Chow group of rank** r. $A^{n-3}(n) \cong \mathbb{Z}$, we denote this isomorphism as $\int : A^{n-3}(n) \longrightarrow \mathbb{Z}$.
- $\{\delta_{I,J} \mid \{I, J\}$ is a cut$\}$ is a set of generators for $A^1(n)$; they are also generators for $A^*(n)$.
- $\prod_{i=1}^{n-3} \delta_{I_i,J_i}$ can be viewed as an element in $A^{n-3}(n)$ since we are in a graded ring.
- Goal: calculate the integral value of this monomial, i.e., $\int(\prod_{i=1}^{n-3} \delta_{I_i,J_i})$.

Keel's quadratic relation, tree monomial

Among the generators of $A^*(n)$, we say the two generators $\delta_{I_1,J_1}, \delta_{I_2,J_2}$ fulfill **Keel's quadratic relation** if the following conditions hold:

- $I_1 \cap I_2 \neq \emptyset$;
- $I_1 \cap J_2 \neq \emptyset$;
- $J_1 \cap I_2 \neq \emptyset$;
- $J_1 \cap J_2 \neq \emptyset$.

And when they are fulfilled, we have $\delta_{I_1,J_1} \cdot \delta_{I_2,J_2} = 0$.

- Hence, if any two factors of the monomial fulfills this relation, the whole integral will be zero.
- We call those monomials where no two factors fulfill this quadratic relation **tree monomial**.
- Since there is a one-to-one correspondence between these monomials and **loaded trees**.

loaded tree, weighted tree, sign of integral value

A **loaded tree with n labels and k edges** is a tree (V, E, h, m), where $h : V \to 2^N$ denotes the labeling function for vertices and $m : E \to \mathbb{N}$ denotes the multiplicity function for edges, such that the following conditions hold:

- $\cup\{h(v)\}_{v\in V, h(v)\neq\emptyset} = N$.
- $\sum_{e\in E} m(e) = k$.
- $\deg(v) + |h(v)| \geq 3$ holds for every $v \in V$.

Remark

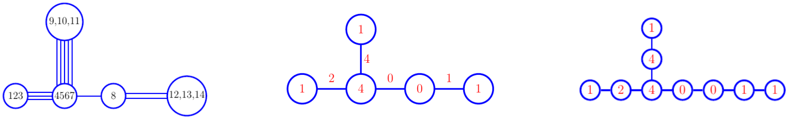
For a loaded tree $LT = (V, E, h, m)$, we define its corresponding **weighted tree** $WT = (V, E, w)$ *by attaching a weight function* $w : V \cup E \to \mathbb{N}$, $e \mapsto m(e) - 1$ *for* $e \in E$; $v \mapsto \deg(v) + |h(v)| - 3$ *for* $v \in V$. *If* $WT = (V, E, w)$ *is* **a weighted tree of some loaded tree with n labels and $n-3$ edges**, *we can easily verify the folowing identity for the weight function:* $\sum_{v\in V} w(v) = \sum_{e\in E} w(e)$. **Given a tree monomial, the sign of its integral value is $(-1)^S$. (S: edge/vertex weight sum)**

one-to-one correspondence, the calculus – running example

Theorem

There is a one-to-one correspondence between tree monomials $T = \prod_{i=1}^{m} \delta_{I_i,J_i}(1 \leq m \leq n-3)$ and loaded trees with n labels and m edges.

We have algorithms converting between loaded trees and tree monomials. Now we illustrate our calculus with an running example. We start from a loaded tree. **Left: loaded tree with 14 labels and 11 edges. Middle: its weighted tree. Right: its redundancy tree.**



the calculus – running example

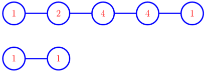
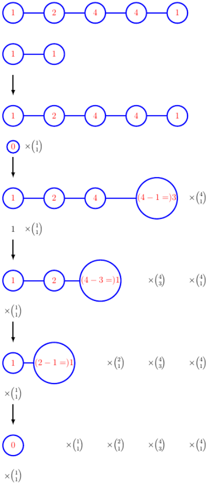
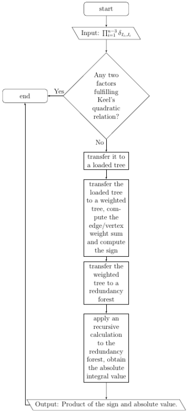


Figure: redundancy forest RF. value of $RF :=$ value of $RF_1 \times \binom{w(\text{unique parent})}{w(\text{leaf})}$ $w_1(\text{unique parent}) := w(\text{unique parent}) - w(\text{leaf})$





Absolute integral value: $1 \times \binom{1}{1} \times \binom{2}{1} \times \binom{4}{3} \times \binom{4}{1} \times \binom{1}{1} = 32.$
Weight sum: 7. Sign: $(-1)^7 = -1$. Final value: -32.

Zoom meeting

- For more discussions, welcome to join our Zoom session from 6 pm (Greek local time), 21st, July.
- https://us04web.zoom.us/j/75988687783?pwd=N3g5YUx1SGpXaU5taFA0VVJKN3lKdz09
- Meeting ID: 759 8868 7783
- Password: 0Lwy3b

Thank You