Introduction to Chow Forms

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2018.05.04.

Background

"The Chow form is a device for assigning invariant 'geometric' coordinates to any subvariety of projective space. It was introduced by Cayley for curves in 3-spaces and later generalized by Chow and van der Waerden."

Several definitions

Definition (Grassmannian)

The set of all d-dimensional linear subspaces of P^n, is called the **Grassmannian** and is denoted by $G(d, n)$.

Definition

In a 3-dimensional projective space P^3, let L be a line through distinct points x and y with homogeneous coordinates $(x_0 : x_1 : x_2 : x_3)$ and $(y_0 : y_1 : y_2 : y_3)$. The **Plücker coordinates** of L p_{ij} are defined as follows:

$$p_{ij} = |x_i, y_i; x_j, y_j| = x_i y_j - x_j y_i.$$

Remark

The definition implies $p_{ii} = 0$ and $p_{ij} = -p_{ji}$, reducing the possibilities to only six independent quantities. The sixtuple is uniquely determined by L up to a common nonzero scale factor.

Dual Plücker coordinates

Alternatively, a line can be described as the intersection of two planes.

Definition

Let L be a line contained in distinct planes **a** and **b** with homogeneous coefficients $(a^0 : a^1 : a^2 : a^3)$ and $(b^0 : b^1 : b^2 : b^3)$,respectively.(The first plane equation is $\sum_K a^K x_K = 0$, for example.) The **dual Plücker coordinate** p^{ij} is

$$p^{ij} = |a^i, a^j; b^i, b^j|.$$

Remark

Dual coordinates are equivalent to primary coordinates:

$$(p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}) = (p^{23} : p^{31} : p^{12} : p^{01} : p^{02} : p^{03}).$$

Here equality means the number on the right side are equal to the numbers on the left side up to some scaling factor λ. *Specially, let* (i, j, k, l) *be an even permutation of* $(0, 1, 2, 3)$*; then*

$$p_{ij} = \lambda p^{kl}.$$

Notations

- X: irreducible projective variety
- $X = \{x \in P^n : f_1(x) = ... = f_r(x) = 0\}$
- f_i are homogeneous polynomials in $k[x_0, ..., x_n]$ and k is a subfield of the complex numbers
- L: (n-d-1)-dimensional linear subspace of P^n
- Y: the set of all (n-d-1)-dimensional linear subspaces L of P^n such that $X \cap L \neq \phi$

Chow form of X

Theorem

The set Y is an irreducible hypersurface in the Grassmannian $G(n-d-1, n)$.

Remark

Every hypersurface in the Grassmannian is defined by a single polynomial equation, the corresponding irreducible polynomial of Y is called the **Chow form** *of X.*

Example: twisted cubic curve

- $X = \{(s^3 : s^2t : st^2 : t^3) \in P^3 : (s : t) \in P^1\}$
- It's the intersection of three quatratic surfaces: $X = \{(x_0 : x_1 : x_2 : x_3) \in P^3 : x_0x_2 - x_1^2 = x_0x_3 - x_1x_2 = x_1x_3 - x_2^2 = 0\}$
- Then we consider a variable line in P^3, $L = kernel(u_0, u_1, u_2, u_3; v_0, v_1, v_2, v_3)$
- Natrually, the line L meets the curve X iff $\exists(s : t) \in P^1$: $u_0s^3 + u_1s^2t + u_2st^2 + u_3t^3 = v_0s^3 + v_1s^2t + v_2st^2 + v_3t^3 = 0$
- this condition is equavalent to the vanishing of **Sylvester resultant**
- Denote $[ij] := u_iv_j - u_jv_i$, then the Sylvester resultant equals Bezout resultant
- so the Chow form of X is $R_X = -[03]^3 - [03]^2[12] + 2[02][03][13] - [01][13]^2 - [02]^2[23] + [01][03][23] + [01][12][23]$

when X is reducible

Definition

If $X = X_1 \cup X_2 \cup \ldots \cup X_r$, where X_i is irreducible and appears m_i times in the above equation, then the Chow form of X is $R_X := R_{X_1}^{m_1} \ldots R_{X_r}^{m_r}$.

Remark

In this case, R_X has coefficient in k, the field of definition of X, while the factors R_{X_i} have coefficients in some algebraic field extension k' of k.

Algorithm: from equations to the Chow form

- Suppose X is an irreducible projective variety, presented by a finite set of generators for the corresponding homogeneous prime ideal $I = I(X)$ in $k[x_0, ..., x_n]$.
- step 0: Compute $d = dim(X)$.
- step 1: Add d+1 linear forms $l_i = u_{i0}x_0 + u_{i1}x_1 + ... + u_{in}x_n$ with indeterminate coefficients, consider the ideal $J = I + \langle l_0, ..., l_d \rangle \subset k[x_i, u_{ji} : i = 0, ..., d, j = 0, ..., n] =: S$.
- step 2: Replace J by $J' = (J : (x_0, ..., x_n)^\infty) = \{f \in S | \forall i \exists d_i : x_i^{d_i} f \in J\}$.
- step 3: Compute the elimination ideal $J' \cap k[u_{00}, u_{01}, ..., u_{dn}]$, since it's in a Euclidean domain, the ideal is principal; let $R(u_{00}, ..., u_{dn})$ be its principal generator.
- step 4: Rewrite the polynomial R in terms of brackets. The result is the Chow form R_X.

Intuition behind

- step 1: form the natural incidence correspondence $\{(x, L) : x \in X \cap L\}$ between X and the Grassmannian.
- step 2: remove trivial solutions with all x-coordinates zero, for which there is no point in P^n.
- step 3 and 4: project the incidence correspondence onto the Grassmannian.

Remark

This algorithm works not only for prime ideals but for all unmixed homogenous ideals.

An example

- Consider the ideal $I = \langle x_1^2 - x_0x_2, x_2^2 - x_1x_3 \rangle$. It's variety consists of the twisted cubic and the line $x_1 = x_2 = 0$.
- add the generic linear forms $u_{00}x_0 + u_{01}x_1 + u_{02}x_2 + u_{03}x_3$ and $u_{10}x_0 + u_{11}x_1 + u_{12}x_2 + u_{13}x_3$.
- saturate it with respect to $\langle x_0, x_1, x_2, x_3 \rangle$ and eliminate the x-variables, then we get a homogeneous polynomial R of bidegree (4,4) in the u_{ij}.
- To obtain a polynomial in brackets, we may take the ideal generated by R together with the polynomial $u_{0i}u_{1j} - u_{0j}u_{1i} - [ij]$ for $0 \leq i < j \leq 3$, and eliminate the u-variables.
- The output is $R_X = [03](-[03]^3 - [03]^2[12] + 2[02][03][13] - [01][13]^2 - [02]^2[23] + [01][03][23] + [01][12][23])$.

From the Chow form to equations

- First rewrite R_X in terms of dual Pl*ü*cker coordinates.
- Then convert them into dual brackets.
- Then substitute each dual bracket $[[ij]]$ with $x_i y_j - x_j y_i$ and expand the result as a polynomial in the y-variables.
- The coefficient polynomials in the x-variables are the Chow equations for X.

An example

- The Chow form of the twisted cubic curve in dual brackets equals: (blackboard)
- Then substitute $[[ij]]$ for $x_i y_j - x_j y_i$, we get
- $$\begin{gathered}(x_3x_2^2 - x_3^2x_1)y_0^2y_2 + (x_2x_3x_1 - x_2^3)y_0^2y_3 + (x_3^2x_1 - x_3x_2^2)y_0y_1^2 \\ + (x_3^2x_0 - x_2x_3x_1)y_0y_1y_2 + (3x_2^2x_1 - 2x_3x_1^2 - x_2x_3x_0)y_0y_1y_3 \\ + (x_3x_1x_0 + 2x_2^2x_0 - 3x_2x_1^2)y_0y_2y_3 + (x_1^3 - x_2x_1x_0)y_0y_3^2 \\ + (x_2^3 - x_3^2x_0)y_1^3 + (3x_2x_3x_0 - 3x_2^2x_1)y_1^2y_2 + (2x_3x_1x_0 - 2x_2^2x_0)y_1^2y_3 \\ + (3x_2x_1^2 - 3x_3x_1x_0)y_1y_2^2 + (x_2x_1x_0 - x_3x_0^2)y_1y_2y_3 + (x_2x_0^2 - x_1^2x_0)y_1y_3^2 \\ + (2x_3x_1^2 - 2x_2x_3x_0)y_0y_2^2 + (x_3x_0^2 - x_1^3)y_2^3 + (x_1^2x_0 - x_2x_0^2)y_2^2y_3\end{gathered}$$
- The coefficients are defining polynomials for the twisted cubic curve, recall: it's $X = \{(x_0 : x_1 : x_2 : x_3) \in P^3 : x_0x_2 - x_1^2 = x_0x_3 - x_1x_2 = x_1x_3 - x_2^2 = 0\}$.

Reference

Dalbec J., Sturmfels B. (1995) Introduction to Chow Forms. In: White N.L. (eds) Invariant Methods in Discrete and Computational Geometry. Springer, Dordrecht

Thank You