Finding rational solutions of rational systems of autonomous ODEs

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Outline

Introduction to the rational autonomous ODEs

Consider the autonomous rational system

$$\begin{cases} s' = \dfrac{N_1(s,t)}{M_1(s,t)} \\ t' = \dfrac{N_2(s,t)}{M_2(s,t)} \end{cases} \tag{1}$$

where $M_1, N_1, M_2, N_2 \in \mathbb{K}[s,t]$, $M_1, M_2 \neq 0$, $\mathbb{K}$ is a field of constants (e.g. $\mathbb{K} = \overline{\mathbb{Q}}$).

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- One can deal with "formal power series solutions", "algebraic solutions", "algebraic integrability", ... of the system.
- We are interested in "rational solutions" of the system, i.e., finding a couple of rational functions $(s(x), t(x)) \in \overline{\mathbb{K}}(x)^2$ satisfying the system.

Motivations and backgrounds to the geometric approach

Why do we study “rational solutions” of the system?

- The problem is interesting by itself.
- We have a machinery of studying “rational algebraic curves”.

Rational algebraic curves

- Given an algebraic curve $F(s,t)=0$. Check the existence of $(s(x),t(x))\in\overline{\mathbb{K}}(x)^2$ with

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- Given a rational parametric curve $(s(x),t(x))$. There is a unique irreducible polynomial $F(s,t)$ such that

$$F(s(x),t(x))=0.$$

  However, the rational parametrization of $F(s,t)=0$ is not unique.

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- We can find a proper rational parametrization $(s(x), t(x))$, i.e. an invertible rational mapping and its inverse is also rational.
- If $(s(x), t(x))$ is a proper rational parametrization of $F(s, t) = 0$ and $(\bar{s}(x), \bar{t}(x))$ is another rational parametrization of $F(s, t) = 0$, then there exists a rational function $T(x)$ such that

$$(\bar{s}(x), \bar{t}(x)) = (s(T(x)), t(T(x))).$$

Autonomous algebraic differential equations

Lemma

Let $F(y, y') = 0$ *be an autonomous algebraic differential equation. Then for every non-trivial rational solution* $y = f(x)$ *of* $F(y, y') = 0$ *we have that*

$$(f(x), f'(x))$$

forms a proper rational parametrization of $F(y, z) = 0$ *and*

$$\deg f = \deg_{y'} F.$$

The ideas of the method

$$\begin{cases} s' = \dfrac{N_1(s,t)}{M_1(s,t)} \\ t' = \dfrac{N_2(s,t)}{M_2(s,t)} \end{cases} \dashrightarrow (s(x), t(x)) \dashrightarrow F(s(x), t(x)) = 0$$

$$\Downarrow$$

$$F_s \cdot N_1 M_2 + F_t \cdot N_2 M_1 = F \cdot K$$

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F is irreducible

$$M_1(s(x), t(x)) \neq 0, M_2(s(x), t(x)) \neq 0 \Downarrow$$

$$\boxed{s'(x) \cdot \frac{N_2(s(x),t(x))}{M_2(s(x),t(x))} = t'(x) \cdot \frac{N_1(s(x),t(x))}{M_1(s(x),t(x))}.}$$

The ideas of the method

$$F_s \cdot N_1 M_2 + F_t \cdot N_2 M_1 = F \cdot K$$

$$\dashleftarrow (s(T(x)), t(T(x))) \dashleftarrow (s(x), t(x)) \dashrightarrow F(s(x), t(x)) = 0$$

proper parametrization

$$\downarrow$$

$$\begin{cases} s' = \dfrac{N_1(s,t)}{M_1(s,t)} \\ t' = \dfrac{N_2(s,t)}{M_2(s,t)} \end{cases} \dashrightarrow T'(x) = \frac{1}{s'(T(x))} \cdot \frac{N_1(s(T(x)), t(T(x)))}{M_1(s(T(x)), t(T(x)))}$$

$$= \frac{1}{t'(T(x))} \cdot \frac{N_2(s(T(x)), t(T(x)))}{M_2(s(T(x)), t(T(x)))}$$

Then

$$(\bar{s}(x), \bar{t}(x)) = (s(T(x)), t(T(x)))$$

is a rational solution.

Claims

1. The rational solutions of the autonomous differential equation

$$T' = \begin{cases} \frac{1}{s'(T)} \cdot \frac{N_1(s(T), t(T))}{M_1(s(T), t(T))} & \text{if } s'(x) \neq 0 \\ \frac{1}{t'(T)} \cdot \frac{N_2(s(T), t(T))}{M_2(s(T), t(T))} & \text{if } t'(x) \neq 0. \end{cases} \tag{2}$$

must be linear rational functions, i.e.,

$$T(x) = \frac{ax + b}{cx + d},$$

where a, b, c and d are constants.

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2. The rational solvability of (2) does not depend on the choice of a proper rational parametrization $(s(x), t(x))$ of $F(s, t) = 0$.

Claims

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   must be linear rational functions, i.e.,

$$T(x) = \frac{ax + b}{cx + d},$$

   where a, b, c and d are constants.

2. The rational solvability of (2) does not depend on the choice of a proper rational parametrization $(s(x), t(x))$ of $F(s, t) = 0$.
3. Every rational solution of the system forms a proper rational parametrization of a rational algebraic curve.

Claims

4.

$$F_s \cdot N_1 M_2 + F_t \cdot N_2 M_1 = F \cdot K.$$

$$F(s(x), t(x)) = 0$$

$$\swarrow \quad \searrow$$

$(s_1(x), t_1(x))$ proper parametrizations $(s_2(x), t_2(x))$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$T_1(x) \qquad\qquad\qquad T_2(x)$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$(s_1(T_1(x)), t_1(T_1(x)))$ solutions $(s_2(T_2(x)), t_2(T_2(x)))$

Then there exists a constant c such that

$$(s_1(T_1(x+c)), t_1(T_1(x+c))) = (s_2(T_2(x)), t_2(T_2(x))).$$

Example

Consider the rational system

$$\begin{cases} s' = \dfrac{-2(t-1)^2(s^2-(t-1)^2)}{((t-1)^2+s^2)^2} \\ t' = \dfrac{-4(t-1)^3 s}{((t-1)^2+s^2)^2}. \end{cases} \tag{3}$$

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Let $d = 2$. The set of irreducible invariant algebraic curves of (3) of degree at most 2 is

$$\{[t-1,\ 2s], [\alpha(t-1)+s,\ \alpha(t-1)+s],$$
$$[-\alpha(t-1)+s,\ -\alpha(t-1)+s],$$
$$[s^2+t^2+(-1-C)t+C,\ 2s]\}$$

where $\alpha = RootOf(Z^2+1)$ and C is an arbitrary constant.

- Consider the line $t - 1 = 0$. It can be parametrized by

$$\mathcal{P}(x) = (x, 1).$$

Then we find a rational function $T(x)$ such that

$$T' = 0.$$

Thus it gives us a solution $\bar{s}(x) = C, \bar{t}(x) = 1$, where C is an arbitrary constant.

- Consider the rational invariant algebraic curve

$$F(s,t) = s^2 + t^2 + (-1 - C)t + C = 0.$$

A proper rational parametrization is

$$\mathcal{P}(x) = \left(\frac{(C-1)x}{1+x^2}, \frac{Cx^2+1}{1+x^2} \right).$$

We find a rational function $T(x)$ such that

$$T' = \frac{-2T^2}{C-1}.$$

Hence

$$T(x) = \frac{C-1}{2x}.$$

Therefore, a rational solution corresponding to $F(s,t) = 0$ is

$$\bar{s}(x) = \frac{2(C-1)^2 x}{4x^2 + (C-1)^2}, \quad \bar{t}(x) = \frac{C(C-1)^2 + 4x^2}{4x^2 + (C-1)^2}.$$

Another example

The system

$$\begin{cases} s' = \dfrac{-2(t-1)^3(-(t-1)^2+s^2)}{((t-1)^2+s^2)^2} \\ t' = \dfrac{-4(t-1)^4 s}{((t-1)^2+s^2)^2}. \end{cases} \tag{4}$$

has no rational solution different from the constant solutions $s(x) = C, t(x) = 1$ because it has the same set of invariant algebraic curves and the autonomous differential equation for the transformation,

$$T' = \frac{-2T^4}{1+T^2},$$

has no rational solution.

Conclusions

1. We have provided a method for finding rational solutions of the differential system (1) using proper parametrizations of rational invariant algebraic curves.
2. We have proven that every rational solution of the differential system (1) forms a proper parametrization for its corresponding rational invariant algebraic curve.

Future works

1. We would like to study rational solutions of some special systems and differential equations.
   e.g.
   $$y' = R(x, y),$$
   where $R(x, y)$ is a rational function in x and y. This is equivalent to looking at the system
   $$\begin{cases} y' = R(x, y) \\ x' = 1. \end{cases}$$
2. Study a degree bound for a rational solution of the differential equation
   $$y' = R(x, y).$$

Thank you for your attention!