Solving parametrizable ODEs

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Outline

Algebraic ODEs of order 1

Definition

An algebraic ordinary differential equation (ODE) of order 1 is given by

$$F(x, y, y') = 0,$$

where

- $F \in \mathbb{K}[x, y, z]$,
- y is an indeterminate over $\mathbb{K}(x)$,
- $y' = \frac{dy}{dx}$,
- $\mathbb{K}$ is a field of constants (algebraically closed field of characteristic 0).

The equation is called autonomous if its coefficients w.r.t. x are zero except for the free coefficient, i.e., $F \in \mathbb{K}[y, z]$.

General solutions of $F(x, y, y') = 0$

A rigorous definition of general solutions of $F(x, y, y') = 0$ can be studied in the framework of differential algebra.

- Differential ring $\mathbb{K}(x)\{y\} = \mathbb{K}(x)[y, y', y'', \ldots]$, $\delta = \frac{d}{dx}$.
- Differential polynomial $F \in \mathbb{K}(x)\{y\}$.
- Differential ideal $[F] =< F, \delta F, \delta^2 F, \ldots >$.
- Radical differential ideal $\{F\} = \sqrt{[F]}$.

We have a decomposition

$$\{F\} = (\{F\} : S) \cap \{F, S\},$$

where S is the separant of F (the partial derivative of F w.r.t the highest derivative appearing in F), i.e., we have

$$\mathcal{Z}(\{F\}) = \mathcal{Z}(\{F\} : S) \cup \mathcal{Z}(\{F, S\}).$$

General solutions of $F(x, y, y') = 0$

Definition

A generic zero of $\{F\} : S$ is called a general solution of $F = 0$, i.e.,

$$\begin{cases} \eta \text{ is a zero of } \{F\} : S, \\ \forall G \in \mathbb{K}(x)\{y\}, G(\eta) = 0 \Longleftrightarrow G \in \{F\} : S. \end{cases}$$

- A rational general solution of $F(x, y, y') = 0$ is a general solution of the form

$$y = \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0},$$

where a_i, b_j are constants in a differential extension field of $\mathbb{K}$.

Autonomous case $F(y, y') = 0$ (R. Feng and X-S. Gao)

Observation: $y = r(x)$ is a non-constant rational solution of $F(y, y') = 0$ if and only if $(r(x), r'(x))$ is a proper parametrization of $F(y, z) = 0$. $\rightsquigarrow$ rational curves

1. compute a proper rational parametrization $(f(x), g(x))$ of $F(y, z) = 0$;
2. compute a rational function $T(x) = \dfrac{ax + b}{cx + d}$ such that

$$f(T(x))' = g(T(x)), \quad \text{i.e.,} \quad T' = \frac{g(T)}{f'(T)};$$

3. `if` there is no such $T(x)$, `then` there is NO rational solution;
4. `else` return the rational general solution

$$y = f(T(x + C))$$

where C is an arbitrary constant.

Extend to parametrizable ODEs

Definition

An algebraic ordinary differential equation $F(x, y, y') = 0$ is called a parametrizable ODE iff the surface $F(x, y, z) = 0$ is rational.

Observation: A non-constant rational solution $r(x)$ of $F(x, y, y') = 0$ is corresponding to the curve parametrized by $(x, r(x), r'(x))$ on the surface $F(x, y, z) = 0$.

Extend to parametrizable ODEs

$F(y, y') = 0$	$F(x, y, y') = 0$
rational curve $F(y, z) = 0$	rational surface $F(x, y, z) = 0$
$\mathcal{P} = (s, f(t), g(t))$	$\mathcal{P} = (\chi_1(s,t), \chi_2(s,t), \chi_3(s,t))$
$\begin{cases} s' = 1 \\ t' = \dfrac{g(t)}{f'(t)} \end{cases}$	$\begin{cases} s' = \dfrac{M_1(s,t)}{N_1(s,t)} \\ t' = \dfrac{M_2(s,t)}{N_2(s,t)} \end{cases} \quad (1)$
$\mathcal{C}(x) = \left(x + C, \dfrac{ax+b}{cx+d}\right)$	$\mathcal{C}(x) = (s(x), t(x))$
$\mathcal{P}(\mathcal{C}(x)) = (x + C, \varphi(x), \varphi'(x))$	$\mathcal{P}(\mathcal{C}(x)) = (x + C, \varphi(x), \varphi'(x))$
$y(x) := f(t(x - C))$	$y(x) := \chi_2(s(x - C), t(x - C))$
$F(y(x), y'(x)) = 0$	$F(x, y(x), y'(x)) = 0$

where $\mathcal{P}(s, t)$ is a proper rational parametrization, C is an arbitrary constant. The system (1) is called the associated system of $F(x, y, y') = 0$ w.r.t $\mathcal{P}(s, t)$.

Associated systems of some special parametrizable ODEs $F(x, y, y') = 0$.

	Solvable for y'	Solvable for y	Solvable for x
ODE	$y' = G(x, y)$	$y = G(x, y')$	$x = G(y, y')$
Surface	$z = G(x, y)$	$y = G(x, z)$	$x = G(y, z)$
Parametrization	$(s, t, G(s, t))$	$(s, G(s, t), t)$	$(G(s, t), s, t)$
A.System	$\begin{cases} s' = 1 \\ t' = G(s, t) \end{cases}$	$\begin{cases} s' = 1 \\ t' = \dfrac{t - G_s(s, t)}{G_t(s, t)} \end{cases}$	$\begin{cases} s' = t \\ t' = \dfrac{1 - tG_s(s, t)}{G_t(s, t)} \end{cases}$

where $G(x, y)$ is a rational function.

Solving the associated system by parametrization method

Associated System	$\begin{cases} s' = \dfrac{M_1(s,t)}{N_1(s,t)} \\ t' = \dfrac{M_2(s,t)}{N_2(s,t)} \end{cases}$
Irr. Inv. Alg. Curve	$G_s \cdot M_1 N_2 + G_t \cdot M_2 N_1 = G \cdot K$
Proper Rat. Para	$(s(x), t(x)),\ \ G(s(x), t(x)) = 0$
Reparametrization	$T' = \dfrac{1}{s'(T)} \cdot \dfrac{M_1(s(T), t(T))}{N_1(s(T), t(T))}$ if $s'(x) \neq 0$ $T' = \dfrac{1}{t'(T)} \cdot \dfrac{M_2(s(T), t(T))}{N_2(s(T), t(T))}$ if $t'(x) \neq 0$ $T(x) = \dfrac{ax+b}{cx+d}$
Rational Solution	$(s(T(x)), t(T(x)))$

Invariant algebraic curves

Definition

A (rational) algebraic curve $G(s, t) = 0$ is called a (rational) invariant algebraic curve of the system (1) iff

$$G_s \cdot M_1 N_2 + G_t \cdot M_2 N_1 = G \cdot K$$

for some polynomial K.

- Computing an irreducible invariant algebraic curve of the system (1) is elementary (i.e., using undetermined coefficients method) provided an upper bound of the degree of the irreducible invariant algebraic curves.
- Such an upper bound is known in a generic case, the case in which the system (1) has no dicritical singularities.

Definition

A rational invariant algebraic curve of the system (1) is called a rational solution curve iff there is a rational parametrization of the curve solving the system.

Theorem

The associated system has a rational general solution corresponding to $G(s,t)=0$ *if and only if* $G(s,t)=0$ *is a rational solution curve and its coefficients contain an arbitrary constant.*

Example 1

Consider the differential equation

$$y'^2 + 3y' - 2y - 3x = 0. \tag{2}$$

It can be parametrized by

$$\mathcal{P}_1(s,t) = \left(\frac{t^2 + 2s + st}{s^2}, -\frac{t^2 + 3s}{s^2}, \frac{t}{s} \right).$$

The associated systems w.r.t. $\mathcal{P}_1(s,t)$ is

$$\begin{cases} s' = st, \\ t' = s + t^2. \end{cases}$$

The irreducible invariant algebraic curves are

$$\{s = 0, t^2 + 2s = 0, cs^2 + t^2 + 2s = 0\},$$

where c is an arbitrary constant.

The rational general solution, corresponding to the curve $cs^2 + t^2 + 2s = 0$, of the associated system is

$$s(x) = -\frac{2}{c + x^2}, \quad t(x) = -\frac{2x}{c + x^2}.$$

Therefore, the rational general solution of (2) is

$$y = \frac{1}{2}((x + c)^2 + 3c).$$

Rational first integrals

Definition

A first integral of the system

$$\begin{cases} s' = \dfrac{M_1(s,t)}{N_1(s,t)}, \\ t' = \dfrac{M_2(s,t)}{N_2(s,t)}, \end{cases}$$

is a non-constant bivariate function $W(s,t)$ such that

$$\frac{M_1}{N_1} \cdot W_s + \frac{M_2}{N_2} \cdot W_t = 0. \tag{3}$$

A first integral $W(s,t)$ of the system (1) is called a rational first integral iff $W(s,t)$ is a rational function in s and t.

Associated System	$\begin{cases} s' = \dfrac{M_1(s,t)}{N_1(s,t)} \\ t' = \dfrac{M_2(s,t)}{N_2(s,t)} \end{cases}$
Rational First Integral	$W = \dfrac{U(s,t)}{V(s,t)},\ \dfrac{M_1}{N_1} \cdot W_s + \dfrac{M_2}{N_2} \cdot W_t = 0$
Factorization in $\overline{\mathbb{K}(c)}[s,t]$ c is a trans. constant	$U - cV = \prod_i (A_i + \alpha_i B_i)$ $U, V, A_i, B_i \in \mathbb{K}[s,t],\ \gcd(U,V) = 1$ $\alpha_i \in \overline{\mathbb{K}(c)}$
Invariant Algebraic Curve	$A_i + \alpha_i B_i = 0,\ \ \forall i$

Rational general solutions and rational first integrals

Theorem

The system (1) has a rational general solution if and only if it has a rational first integral $\dfrac{U}{V} \in \mathbb{K}(s,t)$ *with* $\gcd(U,V) = 1$ *and any irreducible factor of* $U - cV$ *in* $\overline{\mathbb{K}(c)}[s,t]$ *determines a rational solution curve for a transcendental constant* c *over* $\mathbb{K}$*.*

Lemma

The irreducible factors of $U - cV$ *over the field* $\overline{\mathbb{K}(c)}$ *are conjugate over* $\mathbb{K}(c)$ *and they appear in the form*

$$A + \alpha B,$$

where $A, B \in \mathbb{K}[s,t]$ *and* $\alpha \in \overline{\mathbb{K}(c)}$*. Moreover,* α *is also a transcendental constant over* $\mathbb{K}$ *because* c *is so.*

Example 1 (cont.)

In Example 2, a rational first integral of the associated system

$$\begin{cases} s' = st, \\ t' = s + t^2 \end{cases}$$

is

$$W(s,t) = \frac{(t^2 + 2s)^2}{s^4}.$$

We have

$$(t^2 + 2s)^2 - cs^4 = (t^2 + 2s - \sqrt{c}s^2) \cdot (t^2 + 2s + \sqrt{c}s^2).$$

Take $G(s,t) = t^2 + 2s + \sqrt{c}s^2$ as an invariant algebraic curve and proceed as before.

Affine linear transformation on ODEs

(ongoing work with Prof. Rafael Sendra)
Consider the affine linear transformation (birational mapping)

$$\phi(x, y, z) = (x, ay + bx, az + b) \tag{4}$$

and its inverse

$$\phi^{-1}(X, Y, Z) = \left(X, \frac{1}{a}Y - \frac{b}{a}X, \frac{1}{a}Z - \frac{b}{a}\right), \tag{5}$$

where a, b are constants and $a \neq 0$.

- This mapping is compatible with the integral curves on the surfaces $F(x, y, z) = 0$ and $G(X, Y, Z) := F(\phi^{-1}(X, Y, Z)) = 0$, i.e.,

$$(x, f(x), f'(x)) \longmapsto (x, af(x) + bx, af'(x) + b) =: (x, g(x), g'(x)).$$

Theorem

Let $\mathcal{P}(s,t)$ be a proper rational parametrization of $F(x,y,z)=0$. Then $\mathcal{Q}(s,t)=\phi(\mathcal{P}(s,t))$ is a proper rational parametrization of $G(X,Y,Z)$ and the associated system of $G(X,Y,Y')=0$ w.r.t $\mathcal{Q}(s,t)$ is the same as the one of $F(x,y,y')=0$ w.r.t $\mathcal{P}(s,t)$.

Corollary

If $F(x,y,y')=0$ is transformable into an autonomous ODE via the affine change ϕ, then there exists a proper rational parametrization $\mathcal{P}(s,t)$ of $F(x,y,z)=0$ such that its associated system is of the form

$$\begin{cases} s'=1, \\ t'=\dfrac{M(t)}{N(t)}. \end{cases}$$

Affine linear transformation on ODEs - Example

The differential equation

$$y'^2 + 3y' - 2y - 3x = 0$$

is transformable into an autonomous ODE by $y = Y - \frac{3}{2}x$, we obtain

$$Y'^2 - 2Y - \frac{9}{4} = 0.$$

The last equation can be parametrized by $\left(s, \frac{t^2}{2} - \frac{9}{8}, t\right)$. Its associated system is

$$\begin{cases} s' = 1, \\ t' = 1. \end{cases}$$

It suggests to parametrize the first equation by

$$\mathcal{P}_2(s,t) = \left(s, \frac{t^2}{2} - \frac{3}{2}s - \frac{9}{8}, t - \frac{3}{2}\right).$$

Conclusion

1. We solve for rational general solutions of a parametrizable ODE via irreducible invariant algebraic curves of its associated system.
2. We present a relation between rational general solutions of the associated system and its rational first integrals. So we have another algorithmic decision for existence of a rational general solution via rational first integrals of the associated system.
3. We present a class of birational transformations on parametrizable ODEs of order 1 preserving the associated system.

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