Integral Equations and Boundary Value Problems

Exercise, WS 2018/19

Exercise sheet 13 30.01.2019

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73. Consider the Volterra integral equation of first kind
$$\int_0^s x(t)\,dt = f(s)\,, \quad s \in [0,1]\,.$$
Let $f \in C^1([0,1])$ and $f(0) = 0$, compute the general solution in $C([0,1])$. Let now $f(s) := \sinh(s)$ and $f_n^\delta(s) := \sinh(s) + \delta \cdot \sin\left(\frac{ns}{\delta}\right)$, for $\delta > 0$ and $n \in \mathbb{N}, n \geq 2$. Compute analytically the solution and test numerically for different values δ and n. Can you conclude anything from that?

74. Compute
$$\left\| f - f_n^\delta \right\|\,, \qquad \text{and} \qquad \left\| x - x_n^\delta \right\|\,,$$
for f and x from Exercise 73. How does a(n arbitrary) small error in the data δ change the solution?

75. Let $f \in C[a,b]$, $k \in C([a,b]^2)$ continuously differentiable with respect to the second variable und $k(s,s) \neq 0$ for all $s \in [a,b]$. Please show that the Volterra integral equation of first kind
$$\int_a^s k(s,t)x(t)\,dt = f(x)\,, \qquad s \in [a,b]\,,$$
is uniquely solvable, if the solution $y \in C[a,b]$ to the Volterra integral equation of second kind
$$y(s) - \int_a^s \frac{\frac{\partial k(s,t)}{\partial t}}{k(s,s)} y(t)\,dt = \frac{f(s)}{k(s,s)}\,, \qquad s \in [a,b]\,,$$
is continuously differentiable and $y(a) = 0$.
Hint: Try $x(s) := y'(s)$.

76. Please show the reverse of Exercise 75. *Hint:* Try $y(s) := \displaystyle\int_a^s x(t)\,dt$.

77. Let $f \in C^1[0,1]$. Does it hold that
$$\frac{d}{ds}\left(\int_0^1 \frac{f(rs)}{(1-r)^{1-\alpha}}\right) = \int_0^1 \frac{rf'(rs)}{(1-r)^{\alpha-1}}\,?$$
Please explain why this is true or show that it does not always hold.

78. Consider, analogously to Exercise 73, the Volterra integral equation

$$\int_0^s (s-t)x(t)\,dt = f(s)\,, \quad s \in [0,1]$$

for $f(s) := \frac{1}{720}(s^6 - 20s^3 + 45s^2)$. What happens for the "noisy" right hand side $f_n^\delta(s) := f(s) + \delta \cdot \sin\left(\frac{ns}{\delta}\right)$ to the solution x_n^δ of this equation?