Integral Equations and Boundary Value Problems

Exercise, WS 2018/19

Exercise sheet 11 16.01.2019

Dr. Simon Hubmer, S2 503-1, simon.hubmer@ricam.oeaw.ac.at

61. Determine the singular values of the integral operator $K : L^2[0,1] \to L^2[0,1]$ given by

$$(K\phi)(s) = \int_0^s (s-t)\phi(t)\,dt\,, \qquad 0 \le s \le 1\,.$$

What is the inverse of K?

62. Let the kernel k be given by

$$k(s,t) := \begin{cases} s(1-t)\,, & s \le t\,, \\ t(1-s)\,, & s > t\,, \end{cases} \qquad s,t \in [0,1]\,,$$

and let

$$(Kx)(s) := \int_0^1 k(s,t)x(t)\,dt\,.$$

Compute the spectrum of $K : C[0,1] \to C[0,1]$ defined above. Is K positive semi-definite?

Hint: Show that $0 \in \sigma(K)$. Recall that in Exercise 12 you showed that $Kx = y$ is equivalent to $-y'' = x$ with boundary conditions $y(0) = y(1) = 0$. Show and use that the eigenvalue problem $Kx = \lambda x$ is related to the well known Laplace eigenvalue problem $-x'' = \mu x$, $x(0) = x(1) = 0$.

63. For K defined as in Exercise 62, please consider now $K : L^2[0,1] \to L^2[0,1]$. Please show that the Picard condition

$$\sum_{n=1}^{\infty} \frac{|\langle f, x_n\rangle|^2}{\lambda_n^2} < \infty\,,$$

does not hold for $f = s$ but does hold for $f = s(s-1)$ with respect to the eigensystem $(\lambda_n, x_n) := (\frac{1}{n^2\pi^2}, \sqrt{2}\sin(n\pi s))$ of K. What is the meaning of the result for each right-hand side?

64. Please show:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Hint: Use the results of Exercise 62 and Theorem 2.42.

65. Is there a symmetric kernel $k \in L^2[0,1]$, such that the integral operator $K : L^2[0,1] \to L^2[0,1]$ induced by k has the eigenvalues $\lambda_n = \frac{1}{\sqrt{n}}$, $n \in \mathcal{N}$?
Hint: Use Theorem 2.42.

66. Please show that the converse statement of Theorem 2.46 (b) does not hold. In other words, show: if $k \in C([0,1]^2)$ is a symmetric kernel, such that $k(s,s) = 1$ for $s \in [0,1]$; then it may not hold that k induces a positive semi-definite integral operator.
Hint: Try $k(s,t) = 1 - \alpha(s-t)^2$, $s,t \in [0,1]$.