C^2-FINITE SEQUENCES: A COMPUTATIONAL APPROACH

Philipp Nuspl, joint work with A. Jiménez-Pastor, V. Pillwein
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JOHANNES KEPLER
UNIVERSITY LINZ

Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

Overview

- C^2-finite sequences are defined by certain linear recurrence equations.
- We will see how we can compute with them.
- These computations can be done with the package `rec_sequences` in Sage (can be obtained from `github.com/PhilippNuspl/rec_sequences`).
- The package is based on the `ore_algebra` package (Kauers, Jaroschek, and Johansson 2015).

```
sage: from rec_sequences.CFiniteSequenceRing import *
sage: from rec_sequences.C2FiniteSequenceRing import *
```

C-finite sequences

Definition

A sequence $a(n) \in \mathbb{K}^{\mathbb{N}}$ is called C-finite if there are constants $\gamma_0, \ldots, \gamma_r \in \mathbb{K}$, not all zero, such that

$$\gamma_0 a(n) + \cdots + \gamma_{r-1} a(n+r-1) + \gamma_r a(n+r) = 0 \quad \text{for all } n \in \mathbb{N}.$$

- Examples: Fibonacci numbers, Lucas numbers, Pell numbers, etc.
- The set of C-finite sequences is a ring under termwise addition and multiplication.
- Every C-finite sequence can be described by finite amount of data.

```
sage: C = CFiniteSequenceRing(QQ)
sage: f = C([1,1,-1], [1,1])
sage: f[:10]
[1, 1, 2, 3, 5, 8, 13, 21, 34, 55]
```

C^2-finite sequences

Definition

A sequence $a = a(n) \in \mathbb{K}^{\mathbb{N}}$ is called C^2-finite if there are C-finite sequences $c_0(n), \ldots, c_r(n) \in \mathbb{K}^{\mathbb{N}}$ with $c_r(n) \neq 0$ for all $n \in \mathbb{N}$ such that

$$c_0(n)a(n) + \cdots + c_{r-1}(n)a(n+r-1) + c_r(n)a(n+r) = 0 \quad \text{for all } n \in \mathbb{N}.$$

- Contains C- and D-finite (also P-recursive or holonomic) and q-holonomic sequences.
- Describable by finite amount of data.
- Studied by Kotek and Makowsky 2014 and Thanatipanonda and Zhang 2020.

```
sage: C2 = C2FiniteSequenceRing(QQ)
sage: fib_fac = C2([f,-1], [1])
sage: fib_fac[:10] # fibonacci-factorial, A003266
[1, 1, 1, 2, 6, 30, 240, 3120, 65520, 2227680]
```

Skolem-Problem

Skolem-Problem

Does a given C-finite sequence have a zero?

Not known whether decidable in general.

- Decidable for sequences of order ≤ 4 (Ouaknine and Worrell 2012).
- Decidable if we have a unique dominant root (Halava et al. 2005).
- Sometimes the Gerhold-Kauers method using CAD can be applied (Gerhold and Kauers 2005; Kauers and Pillwein 2010).

```
sage: f>0 # use Gerhold-Kauers to show positivity
True
sage: f.has_no_zeros()
True
```

Skolem-Mahler-Lech Theorem

A sequence $(n\,d + r)_{n\in\mathbb{N}}$ for $r, d \in \mathbb{N}$ is called an arithmetic progression.

Skolem-Mahler-Lech Theorem

Let $c(n)$ be C-finite over a field of characteristic 0. Then the set

$$Z_c := \{n \in \mathbb{N} \mid c(n) = 0\}$$

is comprised of a finite set together with a finite number of arithmetic progressions.

```
sage: # A021250, decimal expansion of 1/246
sage: c = C([0,0,0,-1,0,0,0,0,1], [0, 0, 4, 0, 6, 5, 0, 4])
sage: c.zeros()
Zero pattern with finite set {0} and arithmetic progressions:
- Arithmetic progression (5*n+3)_n
- Arithmetic progression (5*n+1)_n
```

Example: Sparse Subsequences

Theorem

Let c be a C-finite sequence. The sequence $c(n^2)$ is C^2-finite.

```
sage: fib_sparse = f.sparse_subsequence(C2) # A054783
sage: fib_sparse
C^2-finite sequence of order 2 and degree 2 with coefficients:
 > c0 (n) : C-finite sequence c0(n): (-1)*c0(n) + (3)*c0(n+1) + (-1)
    *c0(n+2) = 0 and c0(0)=-2 , c0(1)=-5
 > c1 (n) : C-finite sequence c1(n): (-1)*c1(n) + (7)*c1(n+1) + (-1)
    *c1(n+2) = 0 and c1(0)=-3 , c1(1)=-21
 > c2 (n) : C-finite sequence c2(n): (-1)*c2(n) + (3)*c2(n+1) + (-1)
    *c2(n+2) = 0 and c2(0)=1 , c2(1)=2
and initial values a(0)=1 , a(1)=1
sage: fib_sparse[:10]
[1, 1, 5, 55, 1597, 121393, 24157817, 12586269025]
```

Ring

Theorem (Jiménez-Pastor, Nuspl, and Pillwein 2021b)

The set of C^2-finite sequences is a difference ring under termwise addition and multiplication.

Proof idea: Let a, b be C^2-finite. Is $a + b$ a C^2-finite sequence?

- Let R be the smallest $\mathbb{K}$-algebra that contains all coefficients of the recurrences of a, b and their shifts.
- Let $Q(R)$ be its total ring of fractions (localization w.r.t. sequences which do not contain zeros). This ring $Q(R)$ is Noetherian.
- Then,

$$\langle \sigma^n(a+b) \mid n \in \mathbb{N} \rangle_{Q(R)} \subseteq \langle \sigma^n a \mid n \in \mathbb{N} \rangle_{Q(R)} + \langle \sigma^n b \mid n \in \mathbb{N} \rangle_{Q(R)}$$

  is finitely generated.

Computable

- Is the ring computable?
- Algorithm suggested by the previous theorem: Reduce problem of finding a recurrence for $a+b$ to solving a linear system $Ax=b$ over $Q(R)$.
- Not clear how to solve such systems.
- If the zeros of the sequences appearing in the system A can be computed:
  - A solution x can be computed (if such a solution exists).
  - Uses Skolem-Mahler-Lech theorem and Moore-Penrose inverse.
  - Not very efficient.

Example addition

Consider

$$(-1)^n a(n) + a(n+1) = 0, \quad b(n) + b(n+1) = 0, \qquad \text{for all } n \in \mathbb{N}.$$

Ansatz of order 2 for the sequence $a+b$:

$$x_0(n)\,(a(n)+b(n)) + x_1(n)\,(a(n+1)+b(n+1)) + (a(n+2)+b(n+2)) = 0.$$

Using recurrences of a, b this can be written as

$$a(n)\,(x_0(n) - (-1)^n x_1(n) - 1) + b(n)\,(x_0(n) - x_1(n) + 1) = 0.$$

Equating coefficients of a, b to zero yields the linear system

$$\begin{pmatrix} 1 & -(-1)^n \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_0(n) \\ x_1(n) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

which has no solution for even n.

Example addition continued

Ansatz of order 3 yields the linear system

$$\begin{pmatrix} 1 & -(-1)^n & -1 & \\ 1 & -1 & 1 & \end{pmatrix} \begin{pmatrix} x_0(n) \\ x_1(n) \\ x_2(n) \end{pmatrix} = \begin{pmatrix} -(-1)^n \\ 1 \end{pmatrix}.$$

It has the solution

$$(x_0(n), x_1(n), x_2(n)) = \left(\tfrac{1}{2}(-1)^{n+1} + \tfrac{1}{2}, 0, \tfrac{1}{2}(-1)^n + \tfrac{1}{2}\right).$$

Indeed, $c = a + b$ satisfies the recurrence

$$\left(\tfrac{1}{2}(-1)^{n+1} + \tfrac{1}{2}\right) c(n) + \left(\tfrac{1}{2}(-1)^n + \tfrac{1}{2}\right) c(n+2) + c(n+3) = 0,$$

```
sage: var("n")
sage: a = C2([C((-1)^n), 1], [1])
sage: b = C2([1, 1], [1])
sage: c = a+b
sage: c.order(), c.degree()
(3, 2)
```

More closure properties

C^2-finite sequences are also closed under

- partial sums,
- taking subsequences at arithmetic progressions,
- interlacing.

Example

The sequence $\sum_{k=0}^{\lfloor n/3 \rfloor} f((2k+1)^2)$ is C^2-finite.

```
sage: a = fib_sparse.subsequence(2, 1).sum().multiple(3)
sage: a.order(), a.degree()
(9, 147)
```

C^2-finite identities

Let f be the Fibonacci sequence. We denote the fibonomial coefficient by

$$\begin{bmatrix} n \\ k \end{bmatrix}_f = \frac{f(n)f(n-1)\cdots f(n-k+1)}{f(1)\cdots f(k)} = \prod_{i=1}^{n} \frac{f(n-i+1)}{f(i)}.$$

Let l denote the Lucas numbers, then

$$\sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_f = \prod_{k=1}^{n} l(2k),$$

This can be shown using

- q-theory (Kilic, Akkus, and Ohtsuka 2012),
- creative telescoping applied to the C^2-finite case,
- difference rings with idempotent representations (Ablinger and Schneider 2021).

Generating function

Lemma

Let a be C^2-finite and $g(x) = \sum_{n\geq 0} a(n)x^n$ its generating function. Then, g satisfies a functional equation of the form

$$\sum_{k=0}^{m} p_k(x) g^{(d_k)}(\gamma_k x) = p(x)$$

with $p_0, \ldots, p_m, p \in \mathbb{K}[x], d_0, \ldots, d_m \in \mathbb{N}, \gamma_0, \ldots, \gamma_m \in \mathbb{K}$.

- Not all coefficient sequences of functions satisfying such a functional equation are C^2-finite. E.g., not all coefficient sequences of even functions are C^2-finite.

Examples

Example

Let $f(n^2)$ be the sparse subsequence of the Fibonacci sequence f. The generating function g of $f(n^2)$ satisfies the functional equation

$$\begin{aligned}\left(\phi^3 x^2 - \phi^{-3}\right) g\left(\phi^2 x\right) - \left(\psi^3 x^2 - \psi^{-3}\right) g\left(\psi^2 x\right) \\ + xg\left(\phi^4 x\right) - xg\left(\psi^4 x\right) = (\psi - \phi)x + 2(\psi - \phi)\end{aligned}$$

where $\phi := \frac{1+\sqrt{5}}{2}$ denotes the golden ratio and $\psi := \frac{1-\sqrt{5}}{2}$ its conjugate.

```
sage: c = C(2^n+1)
sage: d = C(3^n)
sage: a = C2([c, d], [1])
sage: a.functional_equation()
(x)g(2x) + (x)g(x) + (1/3)g(3x) = 1/3
```

Further generalizations

- D-finite sequences satisfy linear recurrence with polynomial coefficients.
- Can define D^2-finite sequences as sequences satisfying linear recurrence with D-finite coefficients.
- Example: Superfactorial $a(n) = \prod_{k=1}^{n} k!$ (A000178).
- Define C^k-finite (or D^k-finite) sequences as sequences satisfying a linear recurrence with C^{k-1}-finite (or D^{k-1}-finite) coefficients.
- Using the same methods as for C^2-finite: All these are rings (Jiménez-Pastor, Nuspl, and Pillwein 2021a).
- Let c be C-finite. Then, $c(n^k)$ is C^k-finite.

Open problems

- More Examples and counterexamples.
- Asymptotics:
  - Upper bound for a C^2-finite sequence? Conjecture: α^{n^2}.
  - Precise asymptotic behavior (maybe only for subclass of C^2-finite sequences).
- More efficient computations: How can we solve system efficiently?
- Are C^2-finite sequences closed under the Cauchy product?
  - Is the Cauchy product of 2^{n^2} and 3^{n^2} a C^2-finite sequence?

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