ON C^2-FINITE SEQUENCES

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C-finite sequences

Definition

A sequence $c(n) \in \mathbb{K}^{\mathbb{N}}$ is called C-finite if there are constants $\gamma_0, \ldots, \gamma_{r-1} \in \mathbb{K}$ such that

$$\gamma_0 c(n) + \cdots + \gamma_{r-1} c(n+r-1) + c(n+r) = 0 \quad \text{for all } n \in \mathbb{N}.$$

- The sequence $c(n)$ can be described completely by finite amount of data, namely by

$$\gamma_0, \ldots, \gamma_{r-1}, c(0), \ldots, c(r-1).$$

- C-finite sequences form a ring under termwise addition and multiplication. We denote it by $\mathcal{R}_C$.
- Example: Fibonacci-sequence $f(n)$, Lucas numbers, Perrin numbers.

C^2-finite sequences

Definition

A sequence $a = a(n) \in \mathbb{K}^{\mathbb{N}}$ is called C^2-finite if there are C-finite sequences $c_0(n), \dots, c_r(n) \in \mathbb{K}^{\mathbb{N}}$ with $c_r(n) \neq 0$ for all $n \in \mathbb{N}$ such that

$$c_0(n)a(n) + \cdots + c_{r-1}(n)a(n+r-1) + c_r(n)a(n+r) = 0 \quad \text{for all } n \in \mathbb{N}.$$

- The sequence a can again be described completely by finite data.
- Contains C- and D-finite and q-holonomic sequences.
- Similar sequences already studied in
  - Kotek and Makowsky 2014,
  - Thanatipanonda and Zhang 2020.
- Recognizing whether recurrence is valid: Skolem-Problem.

Skolem-Problem

Skolem-Problem

Does a given C-finite sequence have a zero?

Not known whether decidable in general.

- Decidable for small orders (≤ 4), Ouaknine and Worrell 2012.
- Asymptotic analysis can help in many cases.
- CAD can be used to determine sign-pattern of sequence, Gerhold and Kauers 2005.

Examples C^2-finite sequences

Example: Fibonorials (A003266)

Let $f(n)$ be the Fibonacci sequence and $a(n) = \prod_{i=1}^{n} f(i)$. The sequence a is C^2-finite with recurrence

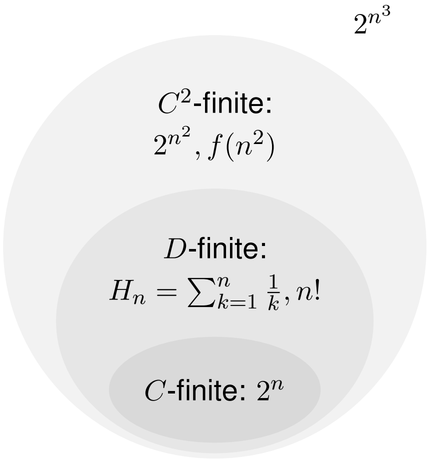
$$f(n+1)a(n) - a(n+1) = 0.$$

They are called fibonorial numbers.

Example: Sparse subsequences

Let $c(n)$ be C-finite. Then, $c(n^2)$ is C^2-finite.

Kotek and Makowsky 2014 give a C^2-finite recurrence for $f(n^2)$ (A054783).



Module of shifts

- $Q(\mathcal{R}_C)$ is the localisation of C-finite sequences w.r.t. the sequences which do not contain any zeros.
- Let $\sigma \colon \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ be the shift operator, i.e. $\sigma\left(a(n)\right) = a(n+1)$.

Theorem

The following are equivalent:

1. The sequence a is C^2-finite
2. The module $\langle \sigma^n a \mid n \in \mathbb{N} \rangle_{Q(\mathcal{R}_C)}$ over the ring $Q(\mathcal{R}_C)$ is finitely generated.

Ring

Let a, b be C^2-finite. Is $a+b$ a C^2-finite sequence?

$$\langle \sigma^n(a+b) \mid n \in \mathbb{N} \rangle_{Q(\mathcal{R}_C)} \subseteq \langle \sigma^n a \mid n \in \mathbb{N} \rangle_{Q(\mathcal{R}_C)} + \langle \sigma^n b \mid n \in \mathbb{N} \rangle_{Q(\mathcal{R}_C)}$$

Submodules of finitely generated modules might not be finitely generated as $\mathcal{R}_C$ is not Noetherian.

Theorem

The set of C^2-finite sequences is a ring under elementwise addition and multiplication.

- Idea: Restrict underlying ring from $\mathcal{R}_C$ to Noetherian subring.
- Order of addition/multiplication depends on coefficients of the C^2-finite sequences.

Addition of C^2-finite sequence

Given C^2-finite a, b of order r_1, r_2. Make ansatz

$$x_0(n)(a(n)+b(n)) + \cdots + x_{s-1}(n)(a(n+s-1)+b(n+s-1)) \\ +(a(n+s)+b(n+s)) = 0$$

of unknown order s and unknown coefficients $x_0, \ldots, x_{s-1} \in Q(\mathcal{R}_C)$. Repeated application of the recurrences and collecting $a(n+i)$ and $b(n+i)$ yields

$$\sum_{i=0}^{r_1-1} \left(\alpha_i(n) + \sum_{j=0}^{s-1} \alpha_{i,j}(n) x_j(n) \right) a(n+i) + \\ \sum_{i=0}^{r_2-1} \left(\beta_i(n) + \sum_{j=0}^{s-1} \beta_{i,j}(n) x_j(n) \right) b(n+i) = 0$$

for some $\alpha_i, \alpha_{i,j}, \beta_i, \beta_{i,j} \in Q(\mathcal{R}_C)$. This equation is certainly true for all n if the coefficient sequences of $a(n+i)$ and $b(n+i)$ are zero.

Addition of C^2-finite sequence

The ansatz yields the linear system

$$Ax = w$$

with given $A \in Q(\mathcal{R}_C)^{(r_1+r_2)\times s}$, $w \in Q(\mathcal{R}_C)^{r_1+r_2}$ and unknown $x \in Q(\mathcal{R}_C)^s$ where the order of the ansatz is denoted by s.

Lemma

If the order of the ansatz s is chosen big enough, then the linear system $Ax = w$ has a solution $x(n) \in \mathbb{K}^s$ for every $n \in \mathbb{N}$.

- Computation of s yields an ideal membership problem in $\mathcal{R}_C$.

Addition of C^2-finite sequence

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Lemma

If a linear system $Ax = w$ has a termwise solution $x(n)$ for every $n \in \mathbb{N}$, then there exists a solution $x \in Q(\mathcal{R}_C)^s$.

- Based on Kotek and Makowsky 2014.
- For computing $x \in Q(\mathcal{R}_C)^s$ we need to solve instances of the Skolem-Problem.
- Hence, the lemma is not fully algorithmic.

Example

Consider

$$((-1)^n + 2^n)\, a(n) - a(n+1) = 0,$$

$$(1 + 2^n)\, b(n) - b(n+1) = 0, \qquad \text{for all } n \in \mathbb{N}.$$

Ansatz of order 2 for the sequence $c = a + b$ yields the equation

$$\begin{pmatrix} 1 & (-1)^n + 2^n \\ 1 & 1 + 2^n \end{pmatrix} \begin{pmatrix} x_0(n) \\ x_1(n) \end{pmatrix} = \begin{pmatrix} -2 \cdot 4^n - (-2)^n + 1 \\ -2 \cdot 4^n - 3 \cdot 2^n - 1 \end{pmatrix}.$$

which has no solution for even n.

Ansatz of order 3 yields recurrence

$$(-2 \cdot 8^n - 4^n + 2 \cdot 2^n - (-1)^n - 2\ (-2)^n + (-4)^n + 2\ (-8)^n + 1)\, c(n)$$

$$(-10 \cdot 4^n - 5 \cdot 2^n + 5\ (-2)^n + 10\ (-4)^n)\, c(n+1)$$

$$(4 \cdot 2^n + (-1)^n + 4\ (-2)^n + 1)\, c(n+2)$$

$$2\, c(n+3) = 0$$

More closure properties

C^2-finite sequences are also closed under

- shifts,
- partial sums,
- taking subsequences at arithmetic progressions,
- interlacing.

Example

Let f denote the Fibonacci-sequence. The sequence

$$\left(\sum_{k=0}^{\lfloor 2n/3 \rfloor} f\left((3k+1)^2\right) \right)_{n \in \mathbb{N}}$$

is C^2-finite.

Fibonomial coefficients

Example: Fibonomial coefficients (Kilic, Akkus, and Ohtsuka 2012)

Let f be the Fibonacci sequence, l the Lucas numbers and

$$\mathrm{Fib}(n,k) := \prod_{i=1}^{k} \frac{f(n-i+1)}{f(k)}$$

the fibonomial coefficient. Then,

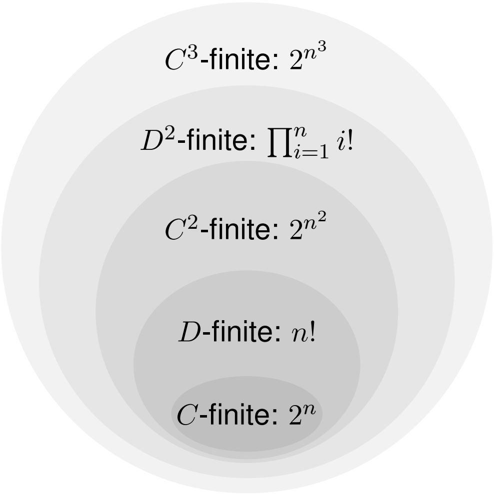
$$\sum_{k=0}^{n} \mathrm{Fib}(2n+1,k) = \prod_{k=1}^{n} l(2k)$$

for all $n \in \mathbb{N}$. In particular, this sequence is C^2-finite.

Identities of this form can be derived and proven fully automatically using difference rings with idempotent representations (Ablinger and Schneider 2021).

D^2 and C^n-finite

- In an analogous way, the set of D^2-finite sequences forms a ring.
- This process can be iterated to show that the sets of C^k and D^k-finite sequences are a ring for all $k \in \mathbb{N}$.
- Jiménez-Pastor and Pillwein 2018, 2019 used a similar construction for functions.



Conclusion

- C^2-finite sequences are a generalization of many well-studied structures.
- They have many closure properties which are usually computable.
- Algorithms can be limited by Skolem-Problem.

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