SIMPLE C^2-FINITE SEQUENCES

A Computable Generalization of C-finite Sequences

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C-finite sequences

Let $\mathbb{K} \supseteq \mathbb{Q}$ be a number field.

Definition

A sequence $c(n) \in \mathbb{K}^{\mathbb{N}}$ is called C-finite if there are constants $\gamma_0, \ldots, \gamma_{r-1} \in \mathbb{K}$ such that

$$\gamma_0 c(n) + \cdots + \gamma_{r-1} c(n+r-1) + c(n+r) = 0 \quad \text{for all } n \in \mathbb{N}.$$

- The sequence $c(n)$ can be described by finite amount of data, namely by
$$\gamma_0, \ldots, \gamma_{r-1}, c(0), \ldots, c(r-1).$$
- C-finite sequences form a computable ring under termwise addition and multiplication.
- Examples: Fibonacci-sequence $f(n)$, Pell numbers, Perrin numbers.

C^2-finite sequences

Definition

A sequence $a = a(n) \in \mathbb{K}^{\mathbb{N}}$ is called C^2-finite if there are C-finite sequences $c_0(n), \ldots, c_r(n) \in \mathbb{K}^{\mathbb{N}}$ with $c_r(n) \neq 0$ for all $n \in \mathbb{N}$ such that

$$c_0(n)a(n) + \cdots + c_{r-1}(n)a(n+r-1) + c_r(n)a(n+r) = 0 \quad \text{for all } n \in \mathbb{N}.$$

- The sequence a can again be described completely by finite data.
- C-, D-finite and q-holonomic sequences are C^2-finite.
- C^2-finite sequences form a ring (Jiménez-Pastor, N., and Pillwein 2021).
- Not clear if the ring is computable.

Skolem problem

Recognizing whether recurrence is valid and computations in C^2-finite sequence ring are limited by Skolem problem:

Skolem problem

Does a given C-finite sequence have a zero?

It is not known whether the problem is decidable in general.

- Decidable for sequences of order ≤ 4 (Ouaknine and Worrell 2012).
- Decidable if we have a unique dominant root (Halava et al. 2005).
- In practice: For "most" sequences it can be checked fully automatically.

Simple C^2-finite sequences

Definition

A sequence $a = a(n) \in \mathbb{K}^{\mathbb{N}}$ is called simple C^2-finite if there are C-finite sequences $c_0(n), \ldots, c_{r-1}(n) \in \mathbb{K}^{\mathbb{N}}$ such that

$$c_0(n)a(n) + \cdots + c_{r-1}(n)a(n+r-1) + a(n+r) = 0 \quad \text{for all } n \in \mathbb{N}.$$

- Most C^2-finite sequences we encountered are simple C^2-finite.
- Catalan numbers are C^2-finite (as they are D-finite) but not simple C^2-finite, cf. Cadilhac et al. 2021.
- For every simple C^2-finite sequence $a(n)$ we can compute an $\alpha \in \mathbb{Q}$ such that $|a(n)| \leq \alpha^{n^2}$ for all $n \geq 1$.

Example

Lemma

Let c be a C-finite sequence. The sequence $\prod_{k=0}^{n} c(k)$ is simple C^2-finite.

In particular, $a(n) = \prod_{k=0}^{n} f(k)$ is simple C^2-finite satisfying

$$-f(n+1)a(n) + a(n+1) = 0.$$

The sequence is called Fibonacci factorial or fibonorials.

Example: Sparse Subsequences

The sequence $f(n^2)$ is C^2-finite satisfying

$$f(2n+3)f(n^2) + f(4n+4)f((n+1)^2) - f(2n+1)f((n+2)^2) = 0.$$

However, $f(n^2)$ is even simple C^2-finite satisfying

$$\begin{aligned} -f(6n+11)f(n^2) \\ -f(4n+6)(-1-2f(4n+4)+3f(4n+6))f((n+1)^2) \\ +f(6n+9)f((n+2)^2) \\ +f((n+3)^2) = 0. \end{aligned}$$

Theorem

Let c be a C-finite sequence. The sequence $c(n^2)$ is simple C^2-finite.

Ring (closure properties)

- Given simple C^2-finite sequences a, b with recurrences
$$c_0(n)a(n) + \cdots + c_{r-1}(n)a(n+r-1) + a(n+r) = 0,$$
$$d_0(n)b(n) + \cdots + d_{s-1}(n)b(n+s-1) + b(n+s) = 0.$$
We want to compute a simple C^2-finite recurrence for $a+b$ and ab.
- Using an ansatz this problem can be reduced to solving linear systems over C-finite sequence ring.
- For C-finite sequences over $\overline{\mathbb{Q}}$ we know how to do this.

Theorem

The set of simple C^2-finite sequences over $\overline{\mathbb{Q}}$ is a computable difference ring.

Example

Consider the simple C^2-finite sequences

$$2^n a(n) + a(n+1) = 0, \quad b(n) + b(n+1) = 0.$$

We want to compute a recurrence for $c = a + b$.

- Using algorithm from (Jiménez-Pastor, N., and Pillwein 2021):

$$\begin{aligned}\left(-2^{5n+4} + 2^{4n+2} + 2^{3n+3} - 2^{2n+1}\right)c(n) \\ + \left(2^{5n+4} - 2^{3n+3} - 2^{2n+1} + 1\right)c(n+2) \\ + \left(2^{4n+2} - 2^{2n+2} + 1\right)c(n+3) = 0.\end{aligned}$$

- Using new algorithm:

$$(2 \cdot 2^n)\, c(n) + (2 + 6 \cdot 2^n)\, c(n+1) + (3 + 4 \cdot 2^n)\, c(n+2) + c(n+3) = 0.$$

Example continued

$$2^n a(n) + a(n+1) = 0, \quad b(n) + b(n+1) = 0.$$

We want to compute a recurrence for $c = a + b$.

- An ansatz of order 3 for c yields the linear system

$$\begin{pmatrix} 1 & -2^n & 2 \cdot 4^n \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \cdot 8^n \\ 1 \end{pmatrix}$$

  for the coefficients $x_0, x_1, x_2 \in \mathbb{Q}^{\mathbb{N}}$ of the recurrence of c.
- Ansatz: $x_i = x_{i,1} + x_{i,2} 2^n$ yields

$$\begin{pmatrix} 1 & -2^n & 2 \cdot 4^n & 2^n & -4^n & 2 \cdot 8^n \\ 1 & -1 & 1 & 2^n & -2^n & 2^n \end{pmatrix} \hat{x} = \begin{pmatrix} 8 \cdot 8^n \\ 1 \end{pmatrix}$$

  where $\hat{x} = (x_{0,1}, x_{1,1}, x_{2,1}, x_{0,2}, x_{1,2}, x_{2,2}) \in \mathbb{Q}^6$.

Example continued

$$2^n a(n) + a(n+1) = 0, \quad b(n) + b(n+1) = 0.$$

We want to compute a recurrence for $c = a + b$.

- We want to solve
$$\begin{pmatrix} 1 & -2^n & 2\cdot 4^n & 2^n & -4^n & 2\cdot 8^n \\ 1 & -1 & 1 & 2^n & -2^n & 2^n \end{pmatrix} \hat{x} = \begin{pmatrix} 8\cdot 8^n \\ 1 \end{pmatrix}.$$
- Comparing the coefficients of $1, 2^n, 4^n, 8^n$ yields the constant system
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix} \hat{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 8 \\ 1 \\ 0 \end{pmatrix}.$$

Example continued

$$2^n a(n) + a(n+1) = 0, \quad b(n) + b(n+1) = 0.$$

We want to compute a recurrence for $c = a + b$.

- Linear system has solution

  $$\hat{x} = (0, 2, 3, 2, 6, 4).$$

  This gives rise to the coefficients x_i of the recurrence for c:

  $$(2 \cdot 2^n)\, c(n) + (2 + 6 \cdot 2^n)\, c(n+1) + (3 + 4 \cdot 2^n)\, c(n+2) + c(n+3) = 0.$$

- If we would not get a solution:
  - ☐ Can increase order of ansatz for c. Then, we solve for x_0, x_1, x_2, x_3.
  - ☐ Or: Can increase ansatz for coefficients x_i. Then, we have the ansatz $x_i = x_{i,1} + x_{i,2} 2^n + x_{i,4} 4^n$.

More closure properties

Theorem

Simple C^2-finite sequences $a(n), b(n)$ are also closed under

- partial sums $\sum_{k=0}^{n} a(k)$,
- taking subsequences at arithmetic progressions $a(ln+k)$ for fixed $l, k \in \mathbb{N}$,
- interlacing

$$(a(0), b(0), a(1), b(1), a(2), b(2), \ldots).$$

Generating functions

Suppose we have a sequence $a \in \mathbb{K}^{\mathbb{N}}$ and we consider its generating function $g(x) = \sum_{n \geq 0} a(n) x^n$.

- a is C-finite iff g is rational.
- a is D-finite (i.e., satisfies a linear recurrence with polynomial coefficient) iff g is D-finite (i.e., satisfies a linear differential equation with polynomial coefficients).
- What kind of equations do the generating functions of (simple) C^2-finite sequences satisfy?
- First ideas were presented in Thanatipanonda and Zhang 2020.

Sequence to generating function

Theorem

Let a be a C^2-finite sequence over $\mathbb{K}$. Let $g(x) = \sum_{n \geq 0} a(n)x^n$ be its generating function. Then, $g(x)$ satisfies a functional equation of the form

$$\sum_{k=1}^{m} p_k(x) g^{(d_k)}(\lambda_k x) = p(x)$$

for $p, p_1, \ldots, p_m \in \mathbb{L}[x], d_1, \ldots, d_m \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{L}$ for some $\mathbb{L} \supseteq \mathbb{K}$.

Let $a(n) = f(n^2)$. The generating function g satisfies

$$\begin{aligned}\left(\phi^3 x^2 - \phi^{-3}\right) g\left(\phi^2 x\right) - \left(\psi^3 x^2 - \psi^{-3}\right) g\left(\psi^2 x\right)& \\ + xg\left(\phi^4 x\right) - xg\left(\psi^4 x\right) &= (\psi - \phi)x\end{aligned}$$

where $\phi := \frac{1+\sqrt{5}}{2}$ denotes the golden ratio and $\psi := \frac{1-\sqrt{5}}{2}$ its conjugate.

Generating function to sequence

Theorem

Let $g(x) = \sum_{n \geq 0} a(n)x^n$ satisfy a functional equation of the form

$$\sum_{k=1}^{m} p_k(x) g^{(d_k)}(\lambda_k x) = p(x)$$

for $p, p_1, \ldots, p_m \in \mathbb{L}[x], d_1, \ldots, d_m \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{L}$. Then, the coefficient sequence $(a(n))_{n \in \mathbb{N}}$ satisfies a linear recurrence with C-finite coefficients over $\mathbb{L}$.

- Not all coefficient sequences of such functions are C^2-finite.
- E.g., $g(x) = g(-x)$ is of the required form, but not all coefficient sequences of even functions are C^2-finite.
- Let $g(x) = \sum_{n \geq 0} a(n)x^n$ satisfy $xg(2x) + g(x) = 1$. Then,

$$2^n a(n) + a(n+1) = 0.$$

Generating functions of simple C^2-finite sequences

Theorem

The sequence $a \in \overline{\mathbb{Q}}^{\mathbb{N}}$ is simple C^2-finite if and only if its generating function $g(x) = \sum_{n \geq 0} a(n) x^n$ satisfies a functional differential equation of the form

$$\sum_{k=1}^{m} \alpha_k x^{j_k} g^{(d_k)}(\lambda_k x) = p(x)$$

for

1. $\alpha_1, \ldots, \alpha_k, \lambda_1, \ldots, \lambda_k \in \overline{\mathbb{Q}} \setminus \{0\}$,
2. $j_1, \ldots, j_m, d_1, \ldots, d_m \in \mathbb{N}$,
3. $p \in \overline{\mathbb{Q}}[x]$ and
4. let $s := \max_{k=1,\ldots,m}(d_k - j_k)$, then for all $k = 1, \ldots, m$ with $d_k - j_k = s$ we have $j_k = 0$ and $\lambda_k = 1$.

Cauchy product

- For (simple) C^2-finite sequences a, b, is the Cauchy product $(a \odot b)(n) := \sum_{i=0}^{n} a(i)b(n-i)$ again (simple) C^2-finite?

Question

Let $a(n) = 2^{n^2}$ and $b(n) = 3^{n^2}$. Is the Cauchy product $a \odot b$ again C^2-finite?

Theorem

Let a be (simple) C^2-finite and b be C-finite. Then, the Cauchy product $a \odot b$ is again (simple) C^2-finite.

Overview

By restricting to simple C^2-finite sequences we obtained:

- a computable ring,
- a computable bound on the growth of the terms,
- an equivalent characterization in terms of the generating function.

Open problems:

- Is the ring of C^2-finite sequences computable?
- Is it possible to derive more precise asymptotic behavior of (simple) C^2-finite sequences?

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