A COMPARISON OF ALGORITHMS FOR PROVING POSITIVITY OF LINEARLY RECURRENT SEQUENCES

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C-finite sequences

Definition

A sequence $c(n) \in \mathbb{Q}^{\mathbb{N}}$ is called C-finite if there are constants $\gamma_0, \ldots, \gamma_{r-1} \in \mathbb{Q}$ such that

$$c(n+r) = \gamma_0 c(n) + \cdots + \gamma_{r-1} c(n+r-1) \quad \text{for all } n \in \mathbb{N}.$$

Examples:

- Fibonacci numbers,
- Pell numbers,
- Perrin numbers.

Problem

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Does $c(n) > 0$ hold for all $n \in \mathbb{N}$?

- In general, it is not known whether the problem is decidable.
- For examples appearing in practice, it usually is decidable (as we will see).
- Which algorithms can be used to prove positivity?

Example (A007910)

Consider the rational function

$$\frac{1}{(1-2x)(1+x^2)} = \sum_{n\geq 0} c(n)x^n.$$

The coefficient sequence $c(n)$ is C-finite satisfying

$$c(n+3) = 2c(n) - c(n+1) + 2c(n+2), \quad c(0) = 1, c(1) = 2, c(2) = 3.$$

Are all coefficients positive, i.e., $c(n) > 0$ for all $n \in \mathbb{N}$?

Theorem (folklore)

A sequence $c(n)$ is C-finite if and only if the generating function $\sum_{n\geq 0} c(n)x^n$ is a rational function.

Gerhold-Kauers method: Example

We have

$$c(n+3) = 2c(n) - c(n+1) + 2c(n+2), \quad c(0) = 1, c(1) = 2, c(2) = 3.$$

- We try to show positivity by induction:

$$(c(n) > 0 \land c(n+1) > 0 \land c(n+2) > 0)$$
$$\implies c(n+3) > 0.$$

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- Let's translate this to formula which can be verified automatically:

$$\forall y_0, y_1, y_2 \in \mathbb{R}\colon\ (y_0 > 0 \land y_1 > 0 \land y_2 > 0) \implies 2y_0 - y_1 + 2y_2 > 0.$$

  Quantifier elimination yields `False`.

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  Quantifier elimination yields `False`.
- Neither proves nor disproves that sequence is positive.

Gerhold-Kauers method: Example

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$$c(n+3) = 2c(n) - c(n+1) + 2c(n+2), \quad c(0) = 1, c(1) = 2, c(2) = 3.$$

- Let's iterate the induction formula:

$$(c(n) > 0 \land c(n+1) > 0 \land c(n+2) > 0 \land c(n+3) > 0)$$
$$\implies c(n+4) > 0.$$

Gerhold-Kauers method: Example

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- Let's iterate the induction formula:

$$(c(n) > 0 \wedge c(n+1) > 0 \wedge c(n+2) > 0 \wedge 2c(n) - c(n+1) + 2c(n+2) > 0)$$
$$\implies 4c(n) + 3c(n+2) > 0.$$

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- The new input for quantifier elimination therefore reads as:

$$\forall y_0, y_1, y_2 \in \mathbb{R}\colon\ (y_0 > 0 \wedge y_1 > 0 \wedge y_2 > 0 \wedge 2y_0 - y_1 + 2y_2 > 0)$$
$$\implies 4y_0 + 3y_2 > 0.$$

Quantifier elimination yields `True`.

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$$\implies 4y_0 + 3y_2 > 0.$$

  Quantifier elimination yields `True`.
- Checking $c(0), \dots, c(3) > 0$ proves that $c(n) > 0$ for all $n \in \mathbb{N}$

Gerhold-Kauers method

- This is known as the Gerhold-Kauers method (Gerhold and Kauers 2005).
- It is not guaranteed to work:
  - If the sequence is not positive, the algorithm will find a counterexample.
  - If the sequence is positive, the algorithm might not terminate (some conditions for termination are known: e.g., Kauers and Pillwein 2010).
- It can be used for other sequences as well (e.g., P-recursive sequences).
- There are variations which can be more powerful.

Closed form

Theorem (folklore)

Let $c(n)$ be C-finite. Then, there is an $n_0 \in \mathbb{N}$ and polynomials $p_1, \dots, p_m \in \overline{\mathbb{Q}}[x]$ and constants $\lambda_1, \dots, \lambda_m \in \overline{\mathbb{Q}}$ such that

$$c(n) = \sum_{i=1}^{m} p_i(n)\lambda_i^n \quad \text{for all } n \geq n_0.$$

We call the λ_i the eigenvalues of the sequence c.

In our example we have

$$c(n) = \tfrac{4}{5}\, 2^n + \left(\tfrac{1}{10} - \tfrac{1}{5}i\right) i^n + \left(\tfrac{1}{10} + \tfrac{1}{5}i\right)(-i)^n \quad \text{for all } n \in \mathbb{N},$$

so the sequence has the eigenvalues $2, i, -i$. Clearly, the sequence will be positive eventually.

Analytic method

We want to show positivity of

$$c(n) = \tfrac{4}{5}\,2^n + \underbrace{\left(\tfrac{1}{10} - \tfrac{1}{5}i\right) i^n + \left(\tfrac{1}{10} + \tfrac{1}{5}i\right) (-i)^n}_{=:r(n)} = \tfrac{4}{5}\,2^n + r(n).$$

Clearly

$$|r(n)| \le |\tfrac{1}{10} - \tfrac{1}{5}i||i|^n + |\tfrac{1}{10} + \tfrac{1}{5}i||-i|^n = \tfrac{1}{\sqrt{5}}.$$

Hence,

$$c(n) = \tfrac{4}{5}\,2^n + r(n) \ge \tfrac{4}{5}\,2^n - |r(n)| = \tfrac{4}{5}\,2^n - \tfrac{1}{\sqrt{5}} > 0$$

for all $n \in \mathbb{N}$, so $c(n)$ is positive.

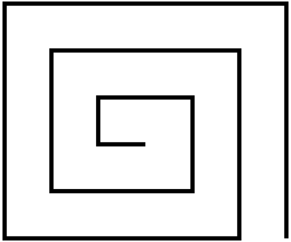
Analytic method

- This method always works if there is a unique dominant eigenvalue, i.e., we have
$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_m|.$$
- Can easily be implemented using arbitrary precision arithmetic or algebraic number arithmetic.
- Analytic method can be extened for sequences with at most 5 dominant eigenvalues (Ouaknine and Worrell 2014).
- For sequences with more than 5 dominant eigenvalues, it is not known whether checking positivity is decidable.

Example 2 (A000969)

Consider the sequence

$$0, 1, 1, 2, 3, 3, 4, 5, 5, 6, 7, 7, 8, 9, 9, 10, 11, 11, \ldots$$



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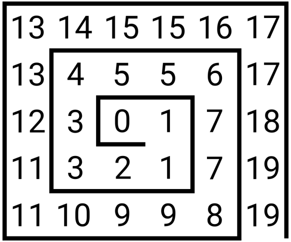

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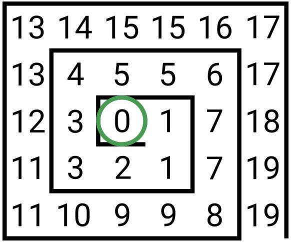

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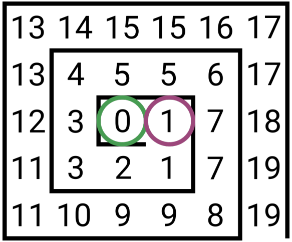

$$1$$

Example 2 (A000969)

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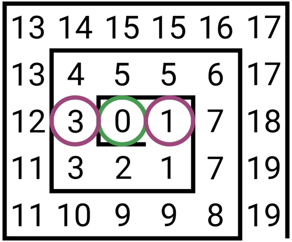
$$0, 1, 1, 2, 3, 3, 4, 5, 5, 6, 7, 7, 8, 9, 9, 10, 11, 11, \ldots$$

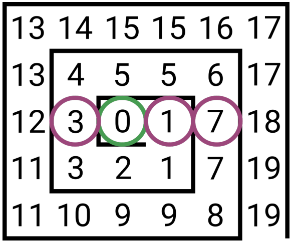


$$1, 3$$

Example 2 (A000969)

Consider the sequence

$$0, 1, 1, 2, 3, 3, 4, 5, 5, 6, 7, 7, 8, 9, 9, 10, 11, 11, \ldots$$

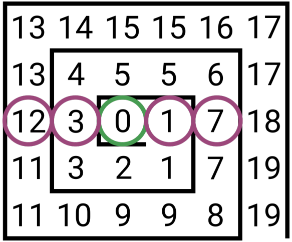


$$1, 3, 7$$

Example 2 (A000969)

Consider the sequence

$$0, 1, 1, 2, 3, 3, 4, 5, 5, 6, 7, 7, 8, 9, 9, 10, 11, 11, \ldots$$

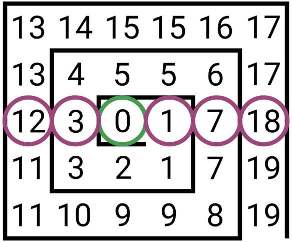


$$1, 3, 7, 12$$

Example 2 (A000969)

Consider the sequence

$$0, 1, 1, 2, 3, 3, 4, 5, 5, 6, 7, 7, 8, 9, 9, 10, 11, 11, \ldots$$

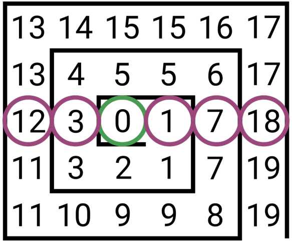


$$1, 3, 7, 12, 18$$

Example 2 (A000969)

Consider the sequence

$$0, 1, 1, 2, 3, 3, 4, 5, 5, 6, 7, 7, 8, 9, 9, 10, 11, 11, \ldots$$



$$1, 3, 7, 12, 18, 26, 35, 45, 57, 70, 84, 100, 117, \ldots$$

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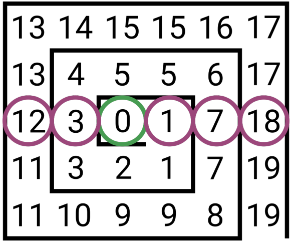
Consider the sequence

$$0, 1, 1, 2, 3, 3, 4, 5, 5, 6, 7, 7, 8, 9, 9, 10, 11, 11, \ldots$$



$$1, 3, 7, 12, 18, 26, 35, 45, 57, 70, 84, 100, 117, \ldots$$

This sequence is C-finite satisfying

$$c(n+5) = c(n) - 2c(n+1) + c(n+2) - c(n+3) + 2c(n+4).$$

Decomposition

We have

$$c(n+5) = c(n) - 2c(n+1) + c(n+2) - c(n+3) + 2c(n+4).$$

- The sequence has the eigenvalues $1, \frac{-1\pm\sqrt{3}\,i}{2}$, the latter being roots of unity.
- Neither the Gerhold-Kauers method nor the analytic method works.
- The subsequences $c(3n), c(3n+1), c(3n+2)$ all have a unique dominant root and we can therefore easily show positivity of all three.
- This gives rise to the positivity of c.
- There is no guarantee that such a decomposition can be found, but usually it works.

Experiments

- We implemented these algorithms (and more) in SageMath and Mathematica.
- Tested them on 1000 positive C-finite sequences from the OEIS with orders

order	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	> 15
	73	134	117	139	120	80	87	36	47	27	31	14	17	10	10	58

- For how many could the SageMath implementation prove positivity with a 60 seconds timeout?

method	Gerhold-Kauers	Analytic	Decomposition
# successfully proven	384	566	984

- Given more time, each of the 1000 sequences could be proven to be positive.

Package: SageMath

Our SageMath package `rec_sequences` provides several methods to prove positivity of C-finite sequences[1]:

```
sage: from rec_sequences.CFiniteSequenceRing import *
sage: C = CFiniteSequenceRing(QQ)
sage: c1 = C([2,-1,2,-1], [1,2,3])
sage: c1 > 0
True
sage: c2 = C([1,-2,1,-1,2,-1], [1,3,7,12,18])
sage: c2 > 0
True
```

[1] It is available at https://github.com/PhilippNuspl/rec_sequences.

Package: Mathematica

For Mathematica our package `PositiveSequence` can be used to prove positivity of C-finite sequences[2]:

In[1]:= $\ll$ **RISC`PositiveSequence`**

In[2]:= $\mathbf{c1 = RE[\{\{0, 2, -1, 2, -1\}, \{1, 2, 3\}\}, c[n]];}$

In[3]:= $\mathbf{PositiveSequence}[\boldsymbol{c}\mathbf{1}]$

Out[3]= True

In[4]:= $\mathbf{c2 = RE[\{\{0, 1, -2, 1, -1, 2, -1\}, \{1, 3, 7, 12, 18\}\}, c[n]];}$

In[5]:= $\mathbf{PositiveSequence}[\boldsymbol{c}\mathbf{2}]$

Out[5]= True

[2] It is available at https://www.risc.jku.at/research/combinat/software/PositiveSequence/.

Conclusions

What have we done?

- Compared several well known and new methods for automatically proving positivity of C-finite sequences.
- Basic methods already cover most sequences encountered in practice.
- Provide implementations in SageMath and Mathematica.

What is left?

- Other, more sophisticated methods are known:
  - Are they more efficient?
  - Do they cover more sequences that appear in practice?
- Methods for P-recursive sequences, i.e., sequences satisfying a linear recurrence with polynomial coefficients.

References

[1] Stefan Gerhold and Manuel Kauers. "A Procedure for Proving Special Function Inequalities Involving a Discrete Parameter". In: Proceedings of ISSAC 2005, Beijing, China, July 24–27, 2005. 2005, pp. 156–162.

[2] Manuel Kauers and Veronika Pillwein. "When Can We Detect That a P-Finite Sequence is Positive?". In: Proceedings of ISSAC 2010, Munich, Germany. New York, NY, USA: Association for Computing Machinery, 2010, pp. 195–201.

[3] Joël Ouaknine and James Worrell. "Positivity problems for low-order linear recurrence sequences". In: SODA '14: Proceedings of the twenty-fifth annual ACM-SIAM symposium on Discrete algorithms. 2014, pp. 366–379.