# The absolute integral value of a sun-like tree

Jiayue Qi

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# background

- Let $n \in \mathbb{N}$, $n \geq 3$, set $N := \{1, \ldots, n\}$.
- A partition $(I, J)$ of $N$ where both cardinality of $I$ and $J$ are at least 2 is called a **cut** (of $M_n$).
- And $I, J$ are called two **parts** of the cut $(I, J)$.
- For each cut $(I, J)$, there is a hypersurface $D_{I,J} \subset M_n$, called *cut subvariety*.
- This talk focus on the Chow ring of $M_n$, where $M_n$ is the moduli space of stable n-pointed curves of genus zero.

# background

- Chow rings are essential in intersection theory, to indicate the intersection numbers of subvarieties.
- Each subvariety has a corresponding element in the Chow ring of the ambient variety.
- Denote $\delta_{I,J}$ as the corresponding element of the cut subvariety $D_{I,J}$ of $M_n$.
- We will not focus on the details of $M_n$, what is important for this talk is the properties of this Chow ring.
- We denote the Chow ring of $M_n$ as $A^*(n)$.

# basic setting

- It is a graded ring, we have $A^*(n) = \bigoplus_{k=0}^{n-3} A^k(n)$; and these homogeneous components are defined as Chow groups (of $M_n$). Here, for instance, we say $A^r(n)$ is a **Chow group of rank** $r$.
- Fact1: $A^r(n) = \{0\}$ for $r > n-3$.
- Fact2: there is a canonical isomorphism $A^{n-3}(n) \cong \mathbb{Z}$ sending the corresponding element of a point to 1, we denote it by $\int : A^{n-3}(n) \longrightarrow \mathbb{Z}$.
- We call $\int(x)$ the *integral value of* $x$ for $x \in A^{n-3}(n)$.
- We extend the defition of $\int$ to the whole ring by defining $\int(x) = 0$ for any $x \notin A^{n-3}(n)$.

## basic setting

- $\{\delta_{I,J} \mid \{I, J\} \text{ is a cut}\}$ is a set of generators for $A^1(n)$; they are also generators for $A^*(n)$, when viewed as ring generators.
- $\prod_{i=1}^{n-3} \delta_{I_i,J_i}$ is an element in $A^{n-3}(n)$.
- Goal: calculate the integral value of this monomial, i.e., $\int(\prod_{i=1}^{n-3} \delta_{I_i,J_i})$.

# motivation

- For me, this calculus shows up as a subproblem when I want to improve an algorithm for realization-counting of Laman graphs on the sphere.
- With the help of the integral value calculation, I invent another algorithm for the same goal.
- However, by efficiency it does not seem faster or better than the existing one.
- But we see that this problem is fundamental, may be helpful for other similar problems, or even further-away problems.
- Then we focus on it, and try to formalize it as a result on its own.

# two important properties of $A^*(n)$

- Quadratic relations between the generators.
- Linear relations between the generators.

# Keel's quadratic relation

We say that the two generators $\delta_{I_1,J_1}, \delta_{I_2,J_2}$ of $A^*(n)$ fulfill **Keel's quadratic relation** if the following conditions hold:

- $I_1 \cap I_2 \neq \emptyset$;
- $I_1 \cap J_2 \neq \emptyset$;
- $J_1 \cap I_2 \neq \emptyset$;
- $J_1 \cap J_2 \neq \emptyset$.

And if so, $\delta_{I_1,J_1} \cdot \delta_{I_2,J_2} = 0$. Easy example: When $n = 5$, $\delta_{12|345} \cdot \delta_{13|245} = 0$ but $\delta_{12|345}$ and $\delta_{123|45}$ does not fulfill this relation.

# Keel's quadratic relation

- Inspired by this property, we know that if any two factors of the monomial fulfills this relation, the whole integral will be zero.
- Now we only need to focus on those monomials where no two factors fulfill this quadratic relation, we call those monomials **tree monomial**.
- This name also has a reason!
- Since there is a one-to-one correspondence between these monomials and a type of tree, which we define as **loaded tree**.

# loaded tree

A **loaded tree with** $n$ **labels and** $k$ **edges** is a tree $(V, E, h, m)$, where $h$ denotes the labeling function from $V$ to the power set of $N$ and $m$ denotes the multiplicity function for edges. The following conditions must hold:

- Non-empty labels $\{h(v)\}_{v \in V}$ form a partition of $N$;
- Number of edges is $k$, edges are counted with multiplicity, i.e., $\sum_{e \in E} m(e) = k$;
- $\deg(v) + |h(v)| \geq 3$ holds for every $v \in V$.

(Hint: this tree would correspond to a monomial in the Chow group $A^k(n)$. )

# loaded tree

See some examples of loaded trees.


Figure: This is a loaded tree with 5 labels and 2 edges.

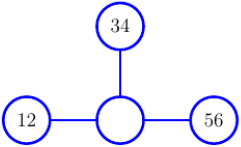



Figure: This is a loaded tree with 6 labels and 3 edges.

# monomial of a given tree

- We define the **monomial of a given loaded tree** as follows:
  - For each edge, when we remove it we get two connected components; we collect the labels in one connected component to form $I$ and labels in the other to form $J$. We say $(I, J)$ is the corresponding cut for this edge.
  - The monomial of this given loaded tree is $\prod_{i=1}^{m} \delta_{I_i, J_i}$, where $m$ is the number of edges.
- Each edge of the tree contributes to the monomial a factor $\delta_{I,J}$ if $(I, J)$ is the corresponding cut for this edge.
- We can see that it is well-defined and each loaded tree has a unique monomial representation.

# monomial of a given tree


Figure: This is a loaded tree with 5 labels and 2 edges, the corresponding tree of tree monomial $\delta_{12|345} \cdot \delta_{123|45}$.

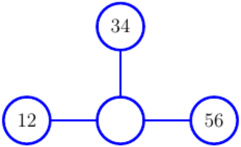



Figure: This is a loaded tree with 6 labels and 3 edges, the corresponding tree of tree monomial $\delta_{34|1256} \cdot \delta_{12|3456} \cdot \delta_{56|1234}$.

# one-to-one correspondence

**Theorem**

*There is a one-to-one correspondence between tree monomials $T = \prod_{i=1}^{m} \delta_{I_i,J_i} (1 \leq m \leq n-3)$ and loaded trees with $n$ labels and $m$ edges, where $I_i \cup J_i = N$ for each $1 \leq i \leq m$. We call the corresponding tree of a tree monomial* **tree of the given tree monomial**.

- We define the *integral value of a loaded tree* as the integral value of its corresponding tree monomial.
- We say a loaded tree is *proper* if its corresponding monomial is in $A^{n-3}(n)$.
- Our focus for this talk is: **calculate the integral value of a sunlike-tree.**
- Before we can introduce the concept of a *sun-like tree*, we need to introduce *weight function* first.

# weight function

- The **weight function** $w : V \cup E \to \mathbb{N}$ of a loaded tree $T = (V, E, h, m)$ is defined as $w(v) := deg(v) + |h(v)| - 3$ for all $v \in V$ and $w(e) := m(e) - 1$ for all $e \in E$.
- It is not hard to verify that $\sum_{v \in V} w(v) = \sum_{e \in E} w(e)$ holds if $T$ is proper.
- We call it the *weight identity*.
- We say a loaded tree is **clever** if all its vertices and edges have value 1.
- The integral value of a clever tree is 1.

# weight function: running examples


Figure: Weights of all vertices and edges are zeroes.

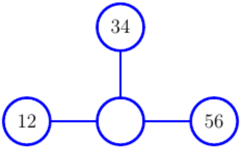
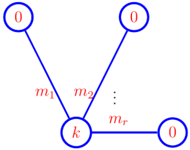



Figure: Weights of the left, upper, middle, right vertices are all zeroes; weights of all three edges are zeroes.

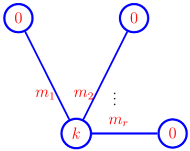
# sun-like tree

- We say a proper loaded tree is **sun-like** if it has domination number equal to one — there exists a vertex $v$ such that all other vertices are neighbors of it — and all adjacent vertices of $v$ have weights zero, all edges have positive weights.
- We call this vertex $v$ the *middle vertex*.
- Let $k$ be the weight for the middle vertex and $m_1, \cdots, m_r \geq 1$ the weights for its incident edges, respectively.
- By the weight identity for proper loaded trees, we know that $k = \sum_{i=1}^{r} m_i$.



# sun-like tree

What is the absolute integral value of a sun-like tree?



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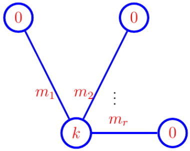


$$\binom{k}{m_1, m_2, \cdots, m_r}$$

# Recall: two important properties of $A^*(n)$

- Quadratic relations between the generators.
- Linear relations between the generators.

# Keel's linear relation

Denote $\epsilon_{ij|kl} := \sum_{i,j\in I,k,l\in J} \delta_{I,J}$. Then we have the equality relations $\epsilon_{ij|kl} = \epsilon_{il|kj} = \epsilon_{ik|jl}$, we call it **Keel's linear relation**.

## Example

When $n = 6$, we have $\epsilon_{12|35} = \epsilon_{13|25} = \epsilon_{15|23}$, i.e.,

$$\delta_{12,3456} + \delta_{124,356} + \delta_{126,345} + \delta_{1246,35}$$

$$= \delta_{13,2456} + \delta_{134,256} + \delta_{136,245} + \delta_{1346,25}$$

$$= \delta_{15,2346} + \delta_{145,236} + \delta_{156,234} + \delta_{1456,23}$$

# Keel's linear relation

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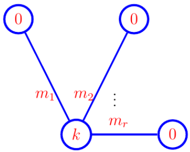
$$= \delta_{15,2346} + \delta_{145,236} + \delta_{156,234} + \delta_{1456,23}$$

## Remark

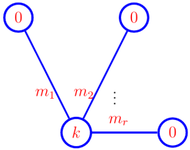
*From the example above we easily see that we can replace some* $\delta_{I,J}$, *say* $\delta_{12|3456}$, *by* $\epsilon_{13|25} - (\epsilon_{12|35} - \delta_{12|3456})$. *Basicly we can replace* $\delta_{I,J}$ *by a sum of* $(2^{n-3} - 1)$ *many* $(\pm)\delta_{I',J'}$.

# value of a sun-like tree




- For the tree $T$ in the above figure, denote the middle vertex by $u$ and the leaf vertices by $v_1, \ldots, v_r$ such that the weight of edge $\{u, v_i\}$ is $m_i$.
- Recall that each edge has a corresponding cut divisor $\delta_{I,J}$.
- Keel's linear relation is equivalent to choosing an edge and a quadruple $(i, j, k, l)$ such that $i, j \in I$ and $k, l \in J$.

# value of a sun-like tree




- Let $M := \delta_1^{d_1} \cdot \ldots \cdot \delta_r^{d_r}$ be the monomial of $T$, where $d_i := m_i + 1$.
- It is not hard to check that $v_1$ has two labels, say $a, b$.
- By definition, number of labels of $u$ — denoted by $\#(u)$ — equals $w(u) - \deg(u) + 3 = \sum_{i=1}^{r} m_i - r + 3 \geq 3$.
- So $u$ has at least three labels, let $c, d$ be two of them.
- We choose $\epsilon_{a,b|c,d} = \epsilon_{a,c|b,d}$ to replace one occurence of $\delta_1 (= \delta_{\{a,b\}})$ in $M$.

# value of a sun-like tree

- By Keel's linear relation,
  $\delta_{ab} = \epsilon_{ac|bd} - \sum_{(I,J) \text{ is a cut, } a,b\in I,\ c,d\in J,\ |I|\geq 3} \delta_{I,J}$.
- For simplicity, denote by
  $\mathcal{S} := \sum_{(I,J) \text{ is a cut, } a,b\in I,\ c,d\in J,\ |I|\geq 3} \delta_{I,J}$.
- We know that $d_1 - 1 = m_1 \leq 1$, and one observes that $\delta_1$ fulfills Keel's quadratic relation with any summand in $\epsilon_{ac|bd}$.
- We get $M = -\delta_1^{m_1} \cdot \delta_2^{d_2} \cdot \ldots \cdot \delta_r^{d_r} \cdot \mathcal{S}$.
- Our next step is to detect-and-remove from $\mathcal{S}$ those summands that fulfills the Keel's quadratic relation with any $\delta_i$.

# value of a sun-like tree

- One can verify that those $\delta_{I,J}$ can "survive" the above process if and only if it fulfills the following condition:
  - $a_i, b_i \in I$ exclusively or $a_i, b_i \in J$ when $i \neq 1$ and $a_1(=a), b_1(=b) \in I$, where $a_i, b_i$ are the labels of $v_i$.
- Let $\mathcal{B}$ be the bi-partition $(\mathcal{B}_1, \mathcal{B}_2)$ of $\{1, \ldots, r\}$ such that $1 \in \mathcal{B}_1$.
- Then

$$
\begin{aligned}
M &= -\delta_1^{d_1-1} \cdot \delta_2^{d_2} \cdot \ldots \cdot \delta_r^{d_r} \cdot \sum_{(\mathcal{B}_1,\mathcal{B}_2)\in\mathcal{B}} \Big( \sum_{I=\{a_i,b_i|i\in\mathcal{B}_1\}} \delta_{I,J} - \delta_1 \Big) \\
&= -\sum_{(\mathcal{B}_1,\mathcal{B}_2)\in\mathcal{B}} \delta_1^{d_1-1} \cdot \delta_2^{d_2} \cdot \ldots \cdot \delta_r^{d_r} \cdot \Big( \sum_{I=\{a_i,b_i|i\in\mathcal{B}_1\}} \delta_{I,J} - \delta_1 \Big)
\end{aligned}
$$

# value of a sun-like tree

- $M = -\sum_{(\mathcal{B}_1,\mathcal{B}_2)\in\mathcal{B}} \delta_1^{d_1-1}\cdot\delta_2^{d_2}\cdot\ldots\cdot\delta_r^{d_r}\cdot\left(\sum_{I=\{a_i,b_i|i\in\mathcal{B}_1\}}\delta_{I,J}-\delta_1\right)$
- Expand the sum in the bracket out, each monomial in the resulting expression is a tree monomial, having a corresponding loaded tree.
- Let us try to find the corresponding tree of any summand in the above mentioned expansion.
- Blackboard.

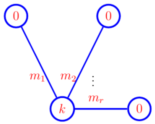
# edge-cutting lemma

- The condition on $I$ in $\mathcal{S}$ guarantees that $\delta_{I,J}$ does not coincide with any $\delta_i$ for $1 \leq i \leq r$. Hence it corresponds to a single edge in the tree.
- Apply the edge-cutting lemma.
- Blackboard.
- Recall that the value of the tree is zero for any tree whose corresponding monomial is not in $A^{n-3}(n)$.
- Therefore the tree value is zero if the number of labels of the tree plus three is unequal to the number of edges.
- Next step is to use this information to detect-and-remove the summands that result in zero.
- Basically we need to consider how to distribute the labels of $u$ to those of $u_1$ and $u_2$, respectively.

## edge-cutting lemma

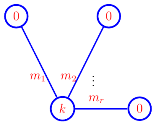
- We know already that $c, d \in h(u_2)$. Hence we can choose from $\#(u) - 2 = k - r + 1$ many labels.
- Let $S(B) := \sum_{i \in B} m_i$, then we know that $w(u_2) = S(\mathcal{B}_2)$.
- By definition, $\#(u_2) + \deg(u_2) - 3 = w(u_2) = S(\mathcal{B}_2)$. So we get $\#(u_2) = S(\mathcal{B}_2) - |\mathcal{B}_2| + 3$.
- However, we know that $c, d$ and the moon label are already fixed for $u_2$. Hence we only need to distribute $S(\mathcal{B}_2) - |\mathcal{B}_2|$ many labels to $u_2$.
- When this distribution law is obeyed, we can have a look at the two smaller trees we got.

## proof sketch




- Denote the absolute value of the above tree as $f(\{m_1, \ldots, m_r\})$, where $f : 2^{\mathbb{N}} \to \mathbb{N}$, note that we allow the domain of $f$ to be multi-sets.
- We define $f(X) := 1$ if $X$ is a set of zeroes, and $f(X \cup \{0\}) := f(X)$. It is not hard to check that this extension of the definition of $f$ also coincides with properties for tree values.

# proof sketch




- Then we have $f(m_1, \ldots, m_r) = \sum_{(\mathcal{B}_1,\mathcal{B}_2)\in\mathcal{B}} \binom{k-r+1}{S(\mathcal{B}_2)-|\mathcal{B}_2|} \cdot f(m_1 - 1, m_i \mid i \in \mathcal{B}_1 \setminus \{1\}) \cdot f(m_j \mid j \in \mathcal{B}_2)$
- One can check that the multinomial coefficient fulfills the base cases for $f$.
- Also, by [1, Theorem 7.2], we see that it also fulfills the recurrence relation listed above.
- Hence, the absolute value of $T$ is $\binom{k}{m_1, m_2, \cdots, m_r}$.

# Reference

Jiayue Qi.
*A tree-based algorithm on monomials in the Chow group of zero cycles in the moduli space of stable pointed curves of genus zero.* arXiv preprint arXiv:2101.03789.

# Thank You