An identity on multinomial coefficients

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DK Seminar

2022.01.11.

motivation

$$\binom{s}{m_1, m_2, \cdots, m_r} = \sum_{(B_1,B_2)\in\mathcal{B}} \binom{s-r+1}{S(B_2)-|B_2|}\binom{S(B_1)}{B_1}\binom{S(B_2)}{B_2},$$

[1, Theorem 7.2].

- The identity showed up when we want to prove the base case for an algorithm computing the integration of monomials in the Chow ring of the moduli space of stable marked curves of genus zero.
- For more background knowledge on where and how the identity showed up, see [1, Section 7].

basic settings

$$\binom{s}{m_1, m_2, \cdots, m_r} = \sum_{(B_1,B_2)\in\mathcal{B}} \binom{s-r+1}{S(B_2)-|B_2|}\binom{S(B_1)}{B_1}\binom{S(B_2)}{B_2}$$

- Let $r \in \mathbb{N}^+$ and $m_1, m_2, \ldots, m_r$ be r-many positive-integer parameters.
- Let $s := \sum_{i=1}^{r} m_i$.
- Denote by $X := \{x_1, x_2, \ldots, x_r\}$ the set of r-many indeterminates.
- Denote by $T := \{B \mid B \subset X,\ x_1 \in B\}$ and by $\mathcal{B} := \{(B_1, B_2) \mid B_1 \in T, B_2 = X \setminus B_1\}$.
- Basically $\mathcal{B}$ is the collection of bipartitions of X, where x_1 is fixed in B_1 and B_2 is allowed to be the emptyset.

basic settings

$$\binom{s}{m_1, m_2, \cdots, m_r} = \sum_{(B_1,B_2)\in\mathcal{B}} \binom{s-r+1}{S(B_2)-|B_2|}\binom{S(B_1)}{B_1}\binom{S(B_2)}{B_2}$$

- Denote by $X := \{x_1, x_2, \ldots, x_r\}$ the set of r-many indeterminates.
- Define a function $g : X \to \{m_1 - 1, m_2, \ldots, m_r\}$ by $g(x_1) := m_1 - 1$ and $g(x_i) = m_i$ when $i \neq 1$.
- Define a "summation function" $S : X \to \mathbb{N}$ by $S(B) := \sum_{x\in B} g(x)$.
- Denote by $\binom{S(B)}{B} := \frac{S(B)!}{\prod_{x\in B}(g(x)!)}$.

An example when $r = 3$: LHS

$$\binom{s}{m_1, m_2, \cdots, m_r} = \sum_{(B_1,B_2)\in\mathcal{B}} \binom{s-r+1}{S(B_2)-|B_2|}\binom{S(B_1)}{B_1}\binom{S(B_2)}{B_2}$$

- For a deeper understanding, let us come to some examples.
- In order to check through the identity with an example, we only need to fix $r \in \mathbb{N}^+$ and $m_1, \ldots, m_r$ r-many positive integers.
- Take $r = 3$, $m_1 = 2$, $m_2 = 3$, $m_3 = 3$, for instance.
- Then $s = \sum_{i=1}^{3} m_i = 2 + 3 + 3 = 8$. Hence LHS $= \binom{s}{m_1, m_2, m_3} = \binom{8}{2,3,3} = 560$.

An example when $r = 3$: RHS

$$\binom{s}{m_1, m_2, \cdots, m_r} = \sum_{(B_1,B_2)\in\mathcal{B}} \binom{s-r+1}{S(B_2)-|B_2|}\binom{S(B_1)}{B_1}\binom{S(B_2)}{B_2}$$

- $\mathcal{B} = \{(\{x_1\},\{x_2,x_3\}),(\{x_1,x_2\},\{x_3\}),(\{x_1,x_3\},\{x_2\}),(\{x_1,x_2,x_3\},\emptyset)\}$.
- Now we need to go through these four elements of $\mathcal{B}$, starting from $B_1 = \{x_1\}$, $B_2 = \{x_2, x_3\}$.
- Then we have $S(B_1) = m_1 - 1 = 2 - 1 = 1$, $S(B_2) = m_2 + m_3 = 3 + 3 = 6$, $\binom{S(B_1)}{B_1} = \frac{S(B_1)!}{g(x_1)!} = \frac{1}{1} = 1$, and $\binom{S(B_2)}{B_2} = \binom{6}{3,3} = 20$.
- $\binom{s-r+1}{S(B_2)-|B_2|} = \binom{8-3+1}{6-2} = \binom{6}{4} = 15$.
- So the corresponding summand for $(\{x_1\},\{x_2,x_3\})$ is $\binom{s-r+1}{S(B_2)-|B_2|}\binom{S(B_1)}{B_1}\binom{S(B_2)}{B_2} = 15 \cdot 1 \cdot 20 = 300$.

An example when $r = 3$: RHS

- Going through the similar process, we obtian the other three summands on RHS as:
  - $\binom{6}{2} \cdot \binom{4}{1,3} = 60$,
  - $\binom{6}{2} \cdot \binom{4}{1,3} = 60$, and
  - $\binom{7}{1,3,3} = 140$.
- Summing up the four summands: $300 + 60 + 60 + 140 = 560$!
- Identity verified for this example.

notations

We slightly modify the notations, so that they serve well for our proof — namely we add an index r for many of them, indicating that we are considering r many sums for the multinomial coefficient.

- $m_1, m_2, \cdots, m_r$: r-many positive-integer parameters.
- $s_r := \sum_{i=1}^{r} m_i$.
- $X_r := \{x_1, x_2, \cdots, x_r\}$: a set of r-many indeterminates.
- $T_r := \{B \mid B \subset X_r, x_1 \in B\}$.
- $\mathcal{B}_r := \{(B_1, B_2) \mid B_1 \in T_r, B_2 = X_r \setminus B_1\}$.
- $g_r : X_r \to \{m_1 - 1, m_2, \cdots, m_r\}$, $x_1 \mapsto m_1 - 1$, $x_i \mapsto m_i$ for $i \neq 1$. This function is introduced for the sake of the next two notations, mainly because the value for m_1 is reduced by one.
- $S(B) := \sum_{x \in B} g_r(x)$, for $B \subset X_r$. This is just the normal sum of m_i for $1 \leq i \leq r$, except that m_1 is replaced by $m_1 - 1$ as a summand — this is also why we need the function g_r.
- $\binom{S(B)}{B} := \frac{S(B)!}{\prod_{x \in B}(g_r(x)!)}$, for $B \subset X_r$.

notations

- Define

$$S_r := \{(P_1, P_2, \cdots, P_r) \mid \cup_{i=1}^r P_i = \{1, 2, \cdots, s_r\},\ |P_i| = m_i\}.$$

  With this set, we collect all partitions of the set $\{1, 2, \cdots, s_r\}$ into r parts such that the i-th part has cardinality m_i.
- Let $L_r := \{2, 3, \cdots, r\}$. These elements are *special* elements in $\{1, 2, \cdots, s_r\}$. Later on we will see why or how they are special, in the definition of the function φ_r. The next two notations are also there to prepare for the definition of the function φ_r.
- For $A \subset \{1, 2, \cdots, r\}$, define $P_A := \cup_{i \in A} P_i$. P_A is the union of the parts which have index in A.
- For $A \subset \{1, 2, \cdots, r\}$, define $X_A := \{x_i \mid i \in A\}$. X_A collect the indeterminates that have index in A.

an example on the notations

Given $r = 3$, the following facts are already clear:

- $X_3 = \{x_1, x_2, x_3\}$.
- $T_3 = \{\{x_1\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_2, x_3\}\}$. This is the collection of one part of the bipartition of X_3 that contains x_1.
- $\mathcal{B}_3 = \{(\{x_1\}, \{x_2, x_3\}), (\{x_1, x_2\}, \{x_3\}), (\{x_1, x_3\}, \{x_2\}), (\{x_1, x_2, x_3\}, \emptyset)\}$.
- $L_3 = \{2, 3\}$. The elements 2 and 3 are special.
- Take $A = \{1, 2\} \subset \{1, 2, 3\}$ for instance, then $X_A = \{x_1, x_2\}$ — the collection of indeterminate with index in A.

an example on the notations

In order to figure out those remaining notations, we should know the values of m_i, $1 \le i \le 3$. Let $m_1 = 2$, $m_2 = 2$ and $m_3 = 1$.

- $s_3 = \sum_{i=1}^{3} m_i = 2+2+1 = 5$. Now we know that $2, 3$ are considered special among $1, 2, 3, 4, 5$.
- $g_3 : X_3 \to \{1, 2\}$ is defined as $g_3(x_1) = m_1 - 1 = 1$, $g_3(x_2) = m_2 = 2$ and $g_3(x_3) = m_3 = 1$.
- Take $B = \{x_2, x_3\} \subset X_3$ for instance, then $S(B) = g_3(x_2) + g_3(x_3) = m_2 + m_3 = 3$.
- Take $B = \{x_2, x_3\} \subset X_3$ for instance, then $\binom{S(B)}{B} = \frac{S(B)!}{\prod_{x \in B} (g_3(x)!)} = \frac{3!}{g_3(x_2) \cdot g_3(x_3)} = \frac{6}{m_2 \cdot m_3} = \frac{6}{2 \cdot 1} = 3$.
- S_3 is the set of all partitions (P_1, P_2, P_3) of the set $\{1, 2, 3, 4, 5\}$ into three parts P_1, P_2, P_3 such that $|P_1| = m_1 = 2$, $|P_2| = m_2 = 2$ and $|P_3| = m_3 = 1$.
- Take $A = \{1, 2\} \subset \{1, 2, 3\}$ for instance, then $P_A = P_1 \cup P_2$ for some $(P_1, P_2, P_3) \in S_3$.

the function ϕ_r

$\phi_r : S_r \to T_r$, $(P_1, \ldots, P_r) \mapsto B \in T_r$.

- Input: $(P_1, \cdots, P_r) \in S_r$.
- Output: $B \in T_r$.
- $B \leftarrow \{x_1\}$
- $A \leftarrow L_r \cap P_1$
- While $A \neq \emptyset$: $B = B \cup X_A$ $A := L_r \cap P_A$.
- Return B.

Check with the example $\varphi_3 : S_3 \to T_3$, $(P_1, P_2, P_3) \mapsto B \in T_3$. $(P_1, P_2, P_3) = (\{1, 3\}, \{4, 5\}, \{2\})$. We see that $L_3 = \{2, 3\}$.

an example on ϕ_r

- Input: $(P_1, P_2, P_3) = (\{1,3\},\{4,5\},\{2\})$.
- Initial values: $B = \{x_1\}$, $A = \{2,3\} \cap \{1,3\} = \{3\}$.
- First loop: Since $A = \{3\} \neq \emptyset$, we have $B = \{x_1\} \cup X_{\{3\}} = \{x_1\} \cup \{x_3\} = \{x_1, x_3\}$, and then $A = \{2,3\} \cap \{2\} = \{2\}$.
- Second loop: Since $A = \{2\} \neq \emptyset$, we have $B = \{x_1, x_3\} \cup X_{\{2\}} = \{x_1, x_3\} \cup \{x_2\} = \{x_1, x_2, x_3\}$, and then $A = \{2,3\} \cap \{4,5\} = \emptyset$.
- Since $A = \emptyset$, return $B = \{x_1, x_2, x_3\}$.
- Output: $B = \{x_1, x_2, x_3\}$.

further analysis using ϕ_r

- $\phi_r : S_r \to T_r$, $(P_1, \dots, P_r) \mapsto B \in T_r$ is a well-defined surjective function.
- Therefore, $\bigcup_{B \in T_r} \varphi_r^{-1}(B) = S_r$, and $|S_r| = \sum_{B \in T_r} |\varphi_r^{-1}(B)| = \sum_{(B, X\setminus B) \in \mathcal{B}_r} |\varphi_r^{-1}(B)|$.
- Recall the identity (with the modified notations): $\binom{s_r}{m_1, m_2, \cdots, m_r} = \sum_{(B_1, B_2) \in \mathcal{B}_r} \binom{s_r - r + 1}{S(B_2) - |B_2|} \binom{S(B_1)}{B_1} \binom{S(B_2)}{B_2}$.
- It remains to show $|\varphi_r^{-1}(B_1)| = \binom{s_r - r + 1}{S(B_2) - |B_2|} \binom{S(B_1)}{B_1} \binom{S(B_2)}{B_2}$.
- In order to prove it, we need to introduce the following result.

a proposition

Proposition

If $\varphi_r(P_1, \cdots, P_r) = B_1$ for some $(P_1, \cdots, P_r) \in S_r$ and $B_1 \in T_r$; denote $B_2 := X_r \setminus B_1$. Then $P_{F_{B_1}} \cap L_r = F_{B_1} \setminus \{1\}$, where $F_B := \{i \mid x_i \in B\}$. Consequently, we have $P_{F_{B_2}} \cap L_r = F_{B_2}$ and $|P_{F_{B_2}} \cap L_r| = |B_2|$.

What this proposition says is the following:

- Given $B_1 \in T_r$ such that $\phi_r(P_1, \ldots, P_r) = B_1$, we know that the special elements in the union of piles defined by B_1 (namely $P_{F_{B_1}}$) form the set $F_{B_1} \setminus \{1\}$.
- And the special elements in the union of piles defined by $B_2 := X_r \setminus B_1$ (namely F_{B_2}) form the set F_{B_2}.

further analysis using the proposition

- Let $K_r := \{1, \cdots, s_r\}$.
- Given $B_1 \in T_r$, we want to find the number of configurations $P := (P_1, \ldots, P_r)$ such that $\phi_r(P) = B_1$.
- We do it in two steps: first decide $P_{F_{B_1}}$ and $P_{F_{B_2}}$, then find out the number of configurations inside these two big groups.
- We know that the special elements are already determined in $P_{F_{B_1}}$ for $(P_1, \ldots, P_r) \in \phi_r^{-1}(B_1)$.
- We only need to choose a proper amount of non-special elements to put in $P_{F_{B_1}}$, i.e. elements in $K_r \setminus L_r$.

further analysis using the proposition

- We need to choose $|P_{F_{B_1}}| - (|B_1| - 1) = (S(B_1) + 1) - |B_1| + 1 = S(B_1) - |B_1| + 2$ many elements from $|K_r \setminus L_r| = s_r - |L_r| = s_r - r + 1$ many elements, and put them in the group of $P_{F_{B_1}}$.
- Since $(S(B_1) - |B_1| + 2) + (S(B_2) - |B_2|) = (S(B_1) + S(B_2) + 1) - (|B_1| + |B_2|) + 1 = s_r - r + 1$, we can also say that there are $\binom{s_r - r + 1}{S(B_2) - |B_2|}$ many ways to arrange the non-special elements.
- This explains the coefficient in the formula:
    - (Recall what we need to show:) $|\varphi_r^{-1}(B_1)| = \binom{s_r - r + 1}{S(B_2) - |B_2|}\binom{S(B_1)}{B_1}\binom{S(B_2)}{B_2}$.

what remains to show

- Recall our strategy: we do it in two steps: first decide $P_{F_{B_1}}$ and $P_{F_{B_2}}$, then find out the number of configurations inside these two big groups.
- Considering the definition of φ_r, we see that no matter how we arrange the elements in $P_{F_{B_2}}$, the image of φ_r is not influenced.
- Therefore, there are $\binom{S(B_2)}{B_2}$ many configurations for the elements in $P_{F_{B_2}}$.
- Recall what we need to show: $|\varphi_r^{-1}(B_1)| = \binom{s_r - r + 1}{S(B_2) - |B_2|}\binom{S(B_1)}{B_1}\binom{S(B_2)}{B_2}$.
- So, we only need to prove that given $B_1 \in T_r$, the number of configurations for the elements in $P_{F_{B_1}}$ is exactly $\binom{S(B_1)}{B_1}$.
- This can be formulated in the proposition below:

the last thing to show

Proposition

Recall that $s_k := \sum_{i=1}^{k} m_i$ *and that* $X_k := \{x_1, \cdots, x_k\}$. *Then we have* $f_k(m_1, m_2, \cdots, m_k) = \binom{s_k - 1}{m_1 - 1, m_2, \cdots, m_k}$, $k \in \mathbb{N}^+$, $m_i \in \mathbb{N}^+$, *where* $f_k : (\mathbb{N}^+)^k \to \mathbb{N}$, $(m_1, m_2, \cdots, m_k) \mapsto |\{(P_1, P_2, \cdots, P_k) \in S_k \mid |P_i| = m_i, \varphi_k(P_1, P_2, \cdots, P_k) = X_k\}|$.

proof of the last proposition

- Prove by two layers of induction.
- When $k = 1$, $L_1 = \emptyset$, for any $m_1 \in \mathbb{N}^+$, we have

$$|\{(P_1) \in S_1 \mid \varphi_1(P_1) = \{x_1\}\}| = 1 = \binom{s_1 - 1}{m_1 - 1}$$

  since $s_1 = m_1$ in this case.
- Assume that the proposition holds whenever the number of parameters is less or equal to $k - 1$, where $k \geq 2$.
- When the number of parameters is k, we start the inner induction on s_k.
- When $s_k = k$, we know that $m_1 = m_2 = \cdots = m_k = 1$.

proof of the last proposition

- Recall how we define ϕ_r, we cannot choose a non-special element for the first pile P_1.
- We can choose any element in L_k for P_1, say i_1; there are $|L_k| = k-1$ many possibilities.
- Then we can choose an element in $L_k \setminus \{i_1\}$ for P_{i_1}, and so on. Until we choose the element $i_{k-1} \in L_k$ for $P_{i_{k-2}}$.
- Then the only remaining part $P_{i_{k-1}}$ can only be $\{1\}$.
- In total there are $(k-1)!$ many configurations.
- Hence we have $f_k(m_1, m_2, \cdots, m_k) = (k-1)! = \binom{k-1}{1,\cdots,1} = \binom{k-1}{0,1,\cdots,1}$, which equals to $\binom{s_k}{m_1-1, m_2, \cdots, m_k}$.

proof of the last proposition

- Assume that the proposition holds whenever the sum of these parameters is less or equal to $s_k - 1$, where we can assume $s_k - 1 \geq k$, i.e., $s_k \geq k + 1$.
- We focus on the position of the element 1 among the piles P_i.
- Since $1 \notin L_k$, it does not influence the image of φ_k on any configuration.
- So in the case when $m_i \geq 2$ for all $1 \leq i \leq k$, there are k-many cases for the distribution of 1:
- $f_k(m_1, m_2, \cdots, m_k) = f_k(m_1 - 1, m_2, \cdots, m_k) + f_k(m_1, m_2 - 1, \cdots, m_k) + \cdots + f_k(m_1, m_2, \cdots, m_k - 1)$.

proof of the last proposition

- Now we can apply the induction hypothesis on the sum of the parameters:
- $f_k(m_1, m_2, \cdots, m_k) = \binom{s_k-2}{m_1-2, m_2, \cdots, m_k} + \binom{s_k-2}{m_1-1, m_2-1, \cdots, m_k} + \cdots + \binom{s_k-2}{m_1-1, m_2, \cdots, m_k-1}$.
- By a known property of multinomial coefficients, we know that the RHS equals $\binom{s_k-1}{m_1-1, m_2, \cdots, m_k}$.
- Hence we get $f_k(m_1, m_2, \cdots, m_k) = \binom{s_k-1}{m_1-1, m_2, \cdots, m_k}$.
- In the case when $m_i = 1$ for some $i \neq 1$, the problem can be reduced to counting the number of corresponding configurations of P_j for $j \neq i$, since $1 \notin L_k$:
- $f_k(m_1, \cdots, m_k) = f_k(m_1 - 1, \cdots, m_k) + \cdots + f_k(m_1, \cdots, m_{i-1} - 1, m_i, \cdots, m_k) + f_{k-1}(m_1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_k) + f_k(m_1, \cdots, m_i, m_{i+1} - 1, \cdots, m_k) + \cdots + f_k(m_1, \cdots, m_{i-1}, m_i, m_{i+1}, \cdots, m_k - 1)$.

proof of the last proposition

- By the outer induction on k, we have:
  $f_{k-1}(m_1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_k) = \binom{(s_k - m_i) - 1}{m_1 - 1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_k} = \binom{(s_k - 1) - 1}{m_1 - 1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_k} = \binom{s_k - 2}{m_1 - 1, \cdots, m_{i-1}, 0, m_{i+1}, \cdots, m_k} = \binom{s_k - 2}{m_1 - 1, \cdots, m_{i-1}, m_i - 1, m_{i+1}, \cdots, m_k}$.
- Substituting it back to the above formula $f_k(m_1, \cdots, m_k) = f_k(m_1 - 1, \cdots, m_k) + \cdots + f_k(m_1, \cdots, m_{i-1} - 1, m_i, \cdots, m_k) + f_{k-1}(m_1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_k) + f_k(m_1, \cdots, m_i, m_{i+1} - 1, \cdots, m_k) + \cdots + f_k(m_1, \cdots, m_{i-1}, m_i, m_{i+1}, \cdots, m_k - 1)$,
- we get $f_k(m_1, m_2, \cdots, m_k) = \binom{s_k - 1}{m_1 - 1, m_2, \cdots, m_k}$.
- With the same idea, it is not hard to prove that the statement holds however many parameters except for m_1 equal(s) one.

proof of the last proposition

- If $m_1 = 1$, from the definition of the function f_k and φ_k, we know that $1 \notin P_1$.
- Hence considering the distribution of the element 1, the recurrence formula becomes: $f_k(m_1, m_2, \cdots, m_k) = f_k(m_1, m_2 - 1, \cdots, m_k) + \cdots + f_k(m_1, m_2, \cdots, m_k - 1)$.
- Then by induction hypothesis on the sum of the parameters, we obtain:
- $f_k(m_1, m_2, \cdots, m_k) = \binom{s_k-2}{m_1-1, m_2-1, \cdots, m_k} + \cdots + \binom{s_k-2}{m_1-1, m_2, \cdots, m_k-1} = \binom{s_k-2}{0, m_2-1, \cdots, m_k} + \cdots + \binom{s_k-2}{0, m_2, \cdots, m_k-1} = \binom{s_k-2}{m_2-1, \cdots, m_k} + \cdots + \binom{s_k-2}{m_2, \cdots, m_k-1} = \binom{s_k-1}{m_2, \cdots, m_k} = \binom{s_k-1}{0, m_2, \cdots, m_k} = \binom{s_k-1}{m_1-1, m_2, \cdots, m_k}$.
- With this, we conclude the proof of the identity.

the identity

$$\binom{s}{m_1, m_2, \cdots, m_r} = \sum_{(B_1,B_2)\in\mathcal{B}} \binom{s-r+1}{S(B_2)-|B_2|} \binom{S(B_1)}{B_1} \binom{S(B_2)}{B_2}$$

Reference

I thank Cristian-Silviu Radu for helping me formulate the identity and as well its proof in a mathematically proper way. The refenrece for this talk is [1, Section 7.5].

Jiayue Qi.
A tree-based algorithm on monomials in the Chow group of zero cycles in the moduli space of stable pointed curves of genus zero. arXiv preprint arXiv:2101.03789.

Thank You